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INVESTIGATION OF THE PERFORMANCE OF VARIOUS MODAL CORRELATION CRITERIA IN STRUCTURAL DAMAGE IDENTIFICATION

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Abstract. *Structural damage identification is a scientific field that has attracted a lot of interest in the scientific community during the recent years. There have been many studies intending to find a reliable method to identify damage in structural elements both in location and extent. Most damage identification methods are based on the changes of dynamic characteristics and static responses, but the incompleteness of the test data is a great obstacle for both. In this paper, the performance of different modal correlation criteria in structural damage identification is investigated. The structural damage identification problem is treated as an optimization problem which is solved using the differential evolution search algorithm. The objective functions used in the optimization process are based on different modal correlation criteria, providing a measure of consistency and correlation between estimations of modal vectors. The performance of each of the objective functions is evaluated by a number of damage scenarios for a simply supported beam. Although the results of the various criteria on the different damage scenarios vary, it is clearly shown that some modal correlation criteria exhibit excellent performance in detecting the structural damage even in the case of strong incompleteness of the modal data.*

1. INTRODUCTION

Structural damage identification has drawn increasing academic literature, as witnessed by the number of relevant journal and conference papers, during the recent years. There have been many studies intending to find a reliable method to identify damage in structural elements both in location and extent [1]. Most damage identification techniques are based on the changes of dynamic characteristics and static responses, but the incompleteness of the test data is a great obstacle for both.

Generally speaking, the existing methods of damage identification techniques based on modal testing can be clarified into two major categories: direct and inverse methods. The direct methods utilize the change in modal measurement to instantly detect structural damage without the need of iterative computational procedures. In contrast, the second category of damage identification techniques poses the whole process as inverse problems [2-9], in which the structural damage is identified via optimizing the correlation between the theoretical and the experimental modal parametric change, respectively. In order to determine the level of correlation between the measured and the predicted natural frequencies or mode shapes modal correlation criteria are used as a simple mathematical tools, providing a measure of consistency and correlation between estimations of modal vectors.

In this paper, the performance of different modal correlation criteria in structural damage identification is investigated. The structural damage identification problem is treated as an optimization problem which is solved using the differential evolution optimization algorithm. The objective functions used in the optimization process are based on different modal correlation criteria to identify the location and the extent of structural damage. The performance of each of the objective functions is evaluated in a number of damage scenarios for a simply supported beam. It is shown that the results of the different modal correlation criteria vary, while certain criteria exhibit excellent performance in detecting the structural damage even in the case of strong incompleteness of the modal data.

2. STRUCTURAL DAMAGE IDENTIFICATION

The problem of damage identification is classified into four levels [10]: (A) detection, (B) localization, (C) quantification, and (D) prediction of future damage (damage prognosis). At the level of damage detection (Level A), the existence of damage can be detected, while its location and severity are unknown. Information about location of the damage can be provided by localization techniques at Level B. At the damage quantification level (Level C), both the location and severity of damage are estimated. Finally, at the prediction level (Level D), the remaining life of the structure is estimated based on the (identified) current damage state and future loads and damage propagation. This study reaches the third level of damage identification, which means it investigates the ability to detect, localize as well as estimate the severity of damage in structures.

It is proven that changes in the dynamic characteristics of a structure are related to damage occurrence. Specifically changes in the modal parameters, namely natural frequencies and mode shapes, can provide an accurate indication of damage in a structure. Since modal parameters are dependent on the physical properties of the structure, i.e. stiffness and mass, the FEM may be used as a tool for locating and quantifying damaged elements in a structure through the update of modal parameters, even in large-scale structures.

2.1. Damage identification model

If a structure is properly modeled using the FEM, structural damage mathematically affects its stiffness and physically its dynamic properties, such as natural frequencies and mode shapes [11]. It can be assumed that the global mass matrix remains the same in both the undamaged and the damaged structure. This assumption is considered quite accurate for the majority of real applications. The eigenvalue problem of a structure with n active degrees of freedom (DOFs) can be written as follows:

$$\left(\mathbf{K}-\omega_{(i)}^{2}\mathbf{M}\right)\left\{\boldsymbol{\varphi}^{(i)}\right\}=0, \quad i=\{1,\ 2,\ldots,\ m\} \tag{1}$$

where $\mathbf{K}$ is the global stiffness matrix of the structure ($[n\times n]$), $\mathbf{M}$ is the global mass matrix ($[n\times n]$), $\{\boldsymbol{\varphi}^{(i)}\}$ is the i-th natural mode vector of the structure ($[n\times 1]$) corresponding to the $\omega_{(i)}$ natural frequency and m is the total number of natural modes to be obtained ($m \leq n$).

Eq. (1) forms the basis of the damage identification method used in the present study. An inverse procedure is used, where the natural frequencies and natural modes of the damaged structure are measured and they are supposed to be known quantities (to a certain extent), while the damage of the structure is unknown and needs to be calculated through an optimization procedure.

2.2. Modal correlation criteria

We consider two structures A and B with n active degrees of freedom (DOFs) each, with eigenvalues $\lambda_{A(i)} = \omega_{A(i)}^2$ and $\lambda_{B(i)} = \omega_{B(i)}^2$, natural frequencies $\omega_{A(i)}$ and $\omega_{B(i)}$ ($i = 1, 2,\ldots, m$), where m is the total number of natural modes obtained ($m \leq n$). The corresponding mode shape vectors are $\{\boldsymbol{\varphi}^{(i)}\}$ and $\{\boldsymbol{\psi}^{(i)}\}$ ($[n\times 1]$ each), for structures A and B, respectively.

In order to compare two sets of values for the two structures, the use of modal correlation criteria is imperative. The following criteria have been used in this study, as useful mathematical tools providing a measure of consistency and correlation between estimations of modal vectors:

1. The Modal Assurance Criterion (MAC)
2. The Modified Total Modal Assurance Criterion (MTMAC)
3. The Co-ordinate Modal Assurance Criterion (CoMAC)
4. The Modal Flexibility Assurance Criterion (MACFLEX)

An example of two structures

In the next sections we will show the mathematical formulation of each criterion and also provide numerical values for a given example of two structures, for better comprehension and in order to exhibit the different criteria used. We consider two example structures, A and B. Each structure has n = 9 active DOFs and up to m = 4 eigenvalues and eigenmodes are taken into consideration (are supposed to be known, for both structures A and B).

Structures A and B correspond to the example which is examined in the numerical results section of the present study. Specifically, Structure A is the structure of the example with no damage, while Structure B is the same structure but in a damaged state where the damage vector is [0.0, 0.0, 0.0, 0.2, 0.3, 0.4, 0.6, 0.6, 0.3, 0.0], i.e. damage 20%, 30%, 40%, 60%, 60%, 30% at the 4th, 5th, 6th, 7th, 8th, 9th element, respectively. The eigenproperties of structures A and B are shown in Table 1 and Table 2. Figure 1 shows the four eigenmodes of the two structures.

		1st Eigenmode	**2nd Eigenmode**	**3rd Eigenmode**	**4th Eigenmode**
	$\lambda = \omega^2$ (sec^{-2})	**3008.56**	**48108.77**	**243219.45**	**765859.94**
	Eigenperiod *T* (sec)	**0.1146**	**0.0286**	**0.0127**	**0.0072**
Nodal values	1st DOF	0.505	-0.960	-1.321	-1.553
	2nd DOF	0.960	-1.553	-1.553	-0.960
	3rd DOF	1.322	-1.553	-0.505	0.960
	4th DOF	1.553	-0.959	0.960	1.553
	5th DOF	1.633	0.001	1.633	0.000
	6th DOF	1.553	0.960	0.960	-1.553
	7th DOF	1.321	1.553	-0.505	-0.960
	8th DOF	0.959	1.553	-1.553	0.960
	9th DOF	0.504	0.960	-1.321	1.553

Table 1. Modal properties (eigenvalues and eigenvectors) of example structure A.

		1st Eigenmode	**2nd Eigenmode**	**3rd Eigenmode**	**4th Eigenmode**
	$\lambda = \omega^2$ (sec^{-2})	**1762.18**	**32163.04**	**174959.58**	**540463.95**
	Eigenperiod *T* (sec)	**0.1497**	**0.0350**	**0.0150**	**0.0085**
Nodal values	1st DOF	0.439	-0.854	-1.206	-1.438
	2nd DOF	0.849	-1.442	-1.552	-1.157
	3rd DOF	1.206	-1.585	-0.780	0.543
	4th DOF	1.476	-1.184	0.636	1.611
	5th DOF	1.627	-0.298	1.631	0.488
	6th DOF	1.629	0.787	1.269	-1.489
	7th DOF	1.441	1.576	-0.366	-1.164
	8th DOF	1.053	1.594	-1.582	1.035
	9th DOF	0.546	0.935	-1.245	1.468

Table 2. Modal properties (eigenvalues and eigenvectors) of example structure B.

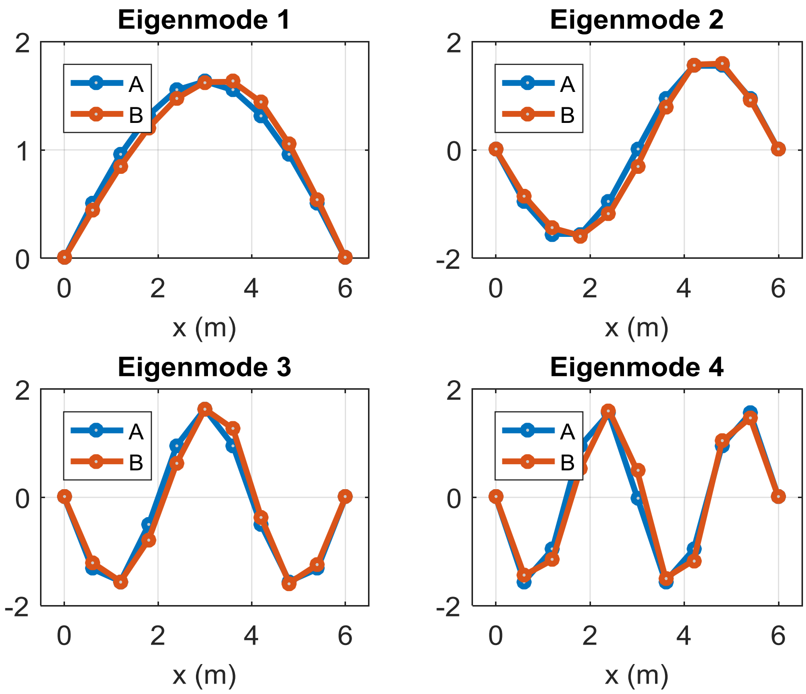


Figure 1. The four eigenmodes of the two example structures A and B.

2.2.1. The Modal Assurance Criterion (MAC)

The Modal Assurance Criterion (MAC) [12, 13] is one of the most popular tools for the quantitative comparison of modal vectors. The purpose of this criterion is to indicate the correlation between two sets of natural modes. Considering two mode shapes vectors $\{\boldsymbol{\varphi}^{(i)}\}$ ($[n\times1]$) and $\{\boldsymbol{\psi}^{(j)}\}$ ($[n\times1]$), for structures A and B, respectively, the term MAC_{ij} of the **MAC** matrix ($[m\times m]$) is given by:

$$MAC_{ij}=\frac{\left(\{\varphi^{(i)}\}^T\{\psi^{(j)}\}\right)^2}{\left(\{\varphi^{(i)}\}^T\{\varphi^{(i)}\}\right)\left(\{\psi^{(j)}\}^T\{\psi^{(j)}\}\right)},\quad i,j=\{1,\ 2,\ldots,\ m\} \tag{2}$$

or

$$MAC_{ij}=\frac{\left(\sum_{k=1}^{n}\varphi_k^{(i)}\psi_k^{(j)}\right)^2}{\sum_{k=1}^{n}\left[\left(\varphi_k^{(i)}\right)^2\right]\cdot\sum_{k=1}^{n}\left[\left(\psi_k^{(j)}\right)^2\right]},\quad i,j=\{1,\ 2,\ldots,\ m\} \tag{3}$$

MAC_{ij} takes values from zero, representing no consistent correspondence, to one, representing a consistent correspondence between the two mode shapes vectors. In this manner, if the modal vectors under consideration, $\{\boldsymbol{\varphi}^{(i)}\}$ and $\{\boldsymbol{\psi}^{(j)}\}$, truly exhibit a consistent relationship, the modal assurance criterion element MAC_{ij} approaches unity. By calculating MAC_{ij} for all $i, j = \{1, 2,\ldots, m\}$ we obtain the **MAC** matrix. In our example, considering all four eigenmodes, we obtain:

$$\mathbf{MAC}(\mathrm{A},\mathrm{A}) = \mathbf{MAC}(\mathrm{B},\mathrm{B}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$\mathbf{MAC}(\mathrm{A},\mathrm{B}) = \mathbf{MAC}(\mathrm{B},\mathrm{A})^T = \begin{bmatrix} 0.9950 & 0.0049 & 0.0001 & 0.0000 \\ 0.0049 & 0.9853 & 0.0077 & 0.0018 \\ 0.0000 & 0.0082 & 0.9765 & 0.0107 \\ 0.0001 & 0.0009 & 0.0121 & 0.9609 \end{bmatrix} \tag{5}$$

If we consider a lower number of eigenmodes m, then the size of the MAC matrix decreases accordingly but in any case, it corresponds to the upper left part of the above full matrix (which corresponds to 4 eigenmodes). In other words, the above matrix contains (as sub-matrices) the [1×1], [2×2], [3×3] **MAC** matrices that would have been calculated for a lower number of known eigenmodes (1, 2 or 3, respectively).

$\mathbf{M\hat{A}C}$ ([1×m]) is a vector holding the diagonal terms of **MAC** matrix and it can be easily calculated by setting $i = j$ in Eq. (2) or Eq. (3). $\mathbf{M\hat{A}C}$ is a vector with as many values as the number of known eigenmodes (m). In our example, considering all four eigenmodes, we obtain:

$$\mathbf{M\hat{A}C}(\mathrm{A},\mathrm{A}) = \mathbf{M\hat{A}C}(\mathrm{B},\mathrm{B}) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \tag{6}$$

$$\mathbf{M\hat{A}C}(\mathrm{A},\mathrm{B}) = \mathbf{M\hat{A}C}(\mathrm{B},\mathrm{A}) = \begin{bmatrix} 0.9950 & 0.9853 & 0.9765 & 0.9609 \end{bmatrix} \tag{7}$$

By multiplying the m individual values of the $\mathbf{M\hat{A}C}$ vector, we obtain the final *MAC* scalar value as follows:

$$MAC = \prod_{i=1}^{m} \mathrm{M\hat{A}C}_i \tag{8}$$

The table below shows the values of *MAC* for various values of the number of known eigenmodes, for our example.

No of known modes	**1**	**2**	**3**	**4**
MAC(A, B)	0.9950	0.9803	0.9573	0.9199

Table 3. *MAC* values for 1, 2, 3 or 4 known eigenmodes.

2.2.2. The modified total modal assurance criterion (MTMAC)

One limitation of the MAC criterion is that it does not take into account the eigenvalues of the different mode shapes of the structures. It takes into account only the eigenvectors, but not the eigenvalues. This means that in case of uniform damage, the MAC criterion will not be able to detect any change, as in this case, the structure becomes more flexible (the eigenperiod increases), but there is no difference in the eigenvectors which remain the same as before. The natural frequencies provide global information of the structure and they can be accurately identified through dynamic measurements.

Another criterion, the modified total modal assurance criterion (MTMAC) [14], is based on the MAC criterion but it takes also the eigenvalues into account. The MTMAC vector $\mathbf{MT\hat{M}AC}$ ($[1\times m]$) is defined as follows:

$$\mathrm{MT\hat{M}AC}_i = \frac{\mathrm{M\hat{A}C}_i}{1+\left|\dfrac{\omega_{A(i)}^2-\omega_{B(i)}^2}{\omega_{A(i)}^2+\omega_{B(i)}^2}\right|}, \quad i=\{1,\ 2,\ldots,\ m\} \tag{9}$$

It should be noted that the MTMAC can be easily defined also as a matrix ($[m\times m]$), whose diagonal is again the $\mathbf{MT\hat{M}AC}$ vector, as was the case with MAC. $\mathbf{MT\hat{M}AC}$ is a row vector with as many values as the number of natural modes considered (m). In our example, considering all four eigenmodes:

$$\mathbf{MT\hat{M}AC}(\mathrm{A},\mathrm{A}) = \mathbf{MT\hat{M}AC}(\mathrm{B},\mathrm{B}) = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} \tag{10}$$

$$\mathbf{MT\hat{M}AC}(\mathrm{A},\mathrm{B}) = \mathbf{MT\hat{M}AC}(\mathrm{B},\mathrm{A}) = \begin{bmatrix}0.7889 & 0.8220 & 0.8395 & 0.8195\end{bmatrix} \tag{11}$$

By multiplying the m individual values of the $\mathbf{MT\hat{M}AC}$ vector, we obtain the *MTMAC* scalar value as follows:

$$MTMAC = \prod_{i=1}^{m} \mathrm{MT\hat{M}AC}_i \tag{12}$$

The table below shows the values of *MTMAC* for various values of the number of known eigenmodes, for our example.

No of known modes	**1**	**2**	**3**	**4**
MTMAC(A, B)	0.7889	0.6484	0.5444	0.4461

Table 4. *MTMAC* values for 1, 2, 3 or 4 known eigenmodes.

2.2.3. The co-ordinate modal assurance criterion (CoMAC)

In the comparison of two sets of modal vectors, one of the issues of interest is the influence of individual DOFs on the vector resemblance. The spatial dependence of the MAC correlation criterion can be misleading. The co-ordinate modal assurance criterion (CoMAC) [15] is an extension of the modal assurance criterion which is used to detect differences between two modal vectors at the DOF level. It is basically a row-wise correlation of two sets of compatible vectors, while in MAC this is done column-wise.

Although we can also define a **COMAC** matrix ($[n\times n]$), in the same way as we defined **MAC** earlier, this time we will go straight to the definition of the $\mathbf{CO\hat{M}AC}$ vector ($[1\times n]$) which is most relevant and important. The off-diagonal terms of **MAC** were the ones giving the relationship between different mode shape vectors of the two structures, for example the element (2, 1) of **MAC** is the one which gives the relationship between the 2nd eigenmode of structure A and the 1st eigenmode of structure B. In the case of **COMAC**, in a similar manner, the off-diagonal terms are the ones giving the relationship between different DOFs of the two structures. For example the element (2, 1) of **COMAC** is the one which gives the relationship between the 2nd DOF of structure A and the 1st DOF of structure B, for the various eigenvectors considered.

Hence, the $\mathbf{CO\hat{M}AC}$ for the k-th DOF of the structure (k = 1, 2,…, n) is defined as follows:

$$\text{CO}\hat{\text{M}}\text{AC}_k = \frac{\left(\sum_{i=1}^{m}\left|\varphi_k^{(i)}\psi_k^{(i)}\right|\right)^2}{\sum_{i=1}^{m}\left[\left(\varphi_k^{(i)}\right)^2\right]\cdot\sum_{i=1}^{m}\left[\left(\psi_k^{(i)}\right)^2\right]}, \quad k=\{1,\ 2,\dots,\ n\} \tag{13}$$

Unlike the MAC, the COMAC can compare modes that are close in frequency by detecting local differences between two sets of modal vectors. It does not identify modeling errors, because their location can be different from the areas where their consequences are felt. Another limitation is the fact that COMAC weights all DOFs equally, irrespective of their magnitude in the modal vector.

By calculating $\text{CO}\hat{\text{M}}\text{AC}_k$ for all k = {1, 2,…, n} we obtain the $\mathbf{CO\hat{M}AC}$ vector ([1×n]). In case of full consistency between $\{\boldsymbol{\varphi}^{(i)}\}$ and $\{\boldsymbol{\psi}^{(i)}\}$ ($\{\boldsymbol{\varphi}^{(i)}\}=\{\boldsymbol{\psi}^{(i)}\}$ for all i = {1, 2,…, m}), all elements of the $\mathbf{CO\hat{M}AC}$ vector will be equal to 1. By multiplying the n individual values of the $\mathbf{CO\hat{M}AC}$ vector, we obtain the *COMAC* scalar value as follows:

$$COMAC = \prod_{i=1}^{n}\text{CO}\hat{\text{M}}\text{AC}_i \tag{14}$$

In our example:

$$\mathbf{CO\hat{M}AC}(\text{A},\text{A}) = \mathbf{CO\hat{M}AC}(\text{B},\text{B}) = \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}^T \tag{15}$$

The table below shows the values of the various elements of $\mathbf{CO\hat{M}AC}$ vector together with the corresponding values of *COMAC* scalar, for various values of the number of known eigenmodes, for our example.

No of known modes	**1**	**2**	**3**	**4**
1st DOF	1.0000	0.9999	0.9997	0.9997
2nd DOF	1.0000	0.9995	0.9979	0.9905
3rd DOF	1.0000	0.9970	0.9804	0.9509
4th DOF	1.0000	0.9850	0.9623	0.9754
5th DOF	1.0000	0.9678	0.9837	0.9421
6th DOF	1.0000	0.9892	0.9760	0.9814
7th DOF	1.0000	0.9987	0.9934	0.9906
8th DOF	1.0000	0.9991	0.9993	0.9992
9th DOF	1.0000	0.9980	0.9983	0.9990
COMAC	**1.0000**	**0.9356**	**0.8955**	**0.8395**

Table 5. $\mathbf{CO\hat{M}AC}$ (A, B) and corresponding *COMAC* scalar value for 1, 2, 3 or 4 known eigenmodes.

2.2.4. The modal flexibility assurance criterion (MACFLEX)

It is well known that damage affects the stiffness matrix of the structure and more specifically it reduces the stiffness of the individual damaged elements. A reduction in stiffness is equivalent to an increase in the structural flexibility.

Flexibility matrix

In structural health monitoring it is advantageous to use changes in flexibility as an indicator of damage rather than using stiffness perturbations. This is due to the following reasons [16]:

1. The flexibility matrix is dominated by the lower modes and so good approximations can be obtained even when only a few lower modes are employed.
2. The flexibility matrices are directly attainable through the modes and mode shapes, determined by the system identification process.
3. Iterative algorithms usually converge the fastest to high eigenvalues.
4. In flexibility-based methods, these eigenvalues correspond to the dominant low-frequency components in structural vibrations.

Therefore, the dynamically measured flexibility matrix which is calculated from the identified modal parameters, can be used as a damage identification measure [7]. The flexibility matrix $\mathbf{F}_{\mathrm{A}}$ ($[n\times n]$) for structure A is given by

$$\mathbf{F}_{\mathrm{A}} = \mathbf{\Phi}\cdot\mathbf{\Lambda}_{\mathrm{A}}^{-1}\cdot\mathbf{\Phi}^{\mathrm{T}} \tag{16}$$

where $\mathbf{\Phi}$ is a matrix ($[n\times m]$) containing all the m mode shape vectors $\{\boldsymbol{\varphi}^{(\mathrm{i})}\}$ ($[n\times 1]$ each) and $\mathbf{\Lambda}_{\mathrm{A}}$ is a diagonal matrix ($[m\times m]$) which holds the eigenvalues $\lambda_{\mathrm{A(i)}} = \omega_{\mathrm{A(i)}}^2$ ($i = 1, 2,\ldots, m$) on its diagonal. The individual elements of matrix $\mathbf{F}_{\mathrm{A}}$ can also be obtained separately using the following formula:

$$F_{A,ij} = \sum_{k=1}^{m} \frac{1}{\omega_{(k)}^2}\varphi_k^{(i)}\varphi_k^{(j)} \tag{17}$$

The two figures below, show graphical representations of the flexibility matrices $\mathbf{F}_{\mathrm{A}}$ and $\mathbf{F}_{\mathrm{B}}$ of the two structures of our example, for 1 and for 4 known eigenmodes, respectively.

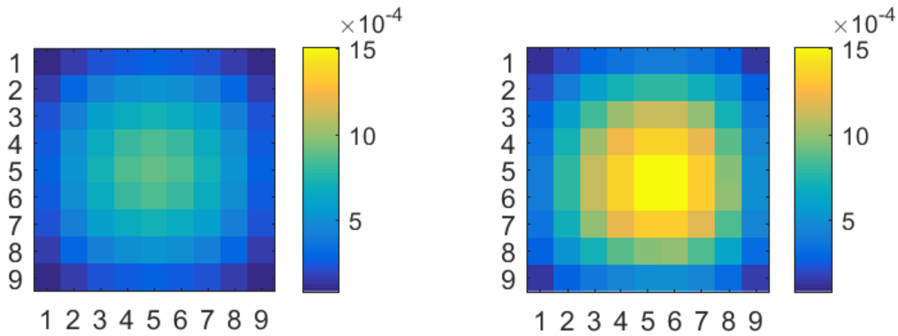


Figure 2. Flexibility matrices for structure A (left) and B (right), for one known eigenmode.

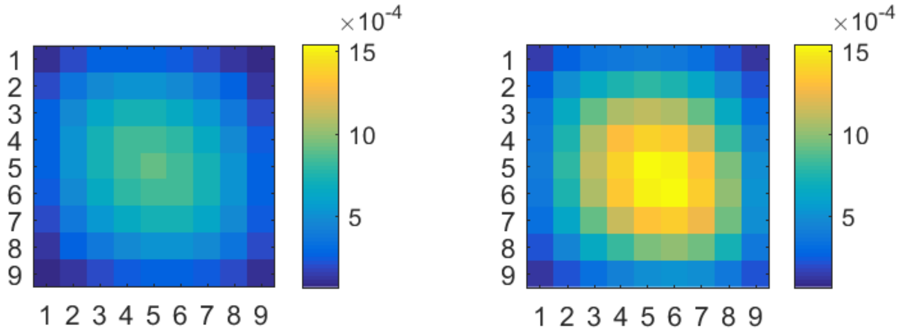


Figure 3. Flexibility matrices for structure A (left) and B (right), for 4 known eigenmodes.

Each column of the flexibility matrix represents the displacement pattern of a structure associated with a unit force applied to the associated degree of freedom. As shown in Eq. (17), as the value of frequency decreases (i.e. the eigenperiod increases) the modal contribution to the flexibility matrix increases also. As a result, a good estimate of the flexibility matrix can be calculated with a small number of the first low-frequency modes, which is also evidenced in the two figures above.

The MACFLEX criterion definition

In order to compare the values of the flexibility matrix of the two structures A and B, the modal flexibility assurance criterion (MACFLEX) is applied. The individual elements of the $\mathbf{MAC\hat{F}LEX}$ vector ([1×n]) can be calculated as follows:

$$\mathrm{MAC\hat{F}LEX}_i = \frac{\left(\hat{\mathbf{F}}_{\mathrm{A}}^{(i)T}\hat{\mathbf{F}}_{\mathrm{B}}^{(i)}\right)^2}{\left(\hat{\mathbf{F}}_{\mathrm{A}}^{(i)T}\hat{\mathbf{F}}_{\mathrm{A}}^{(i)}\right)\left(\hat{\mathbf{F}}_{\mathrm{B}}^{(i)T}\hat{\mathbf{F}}_{\mathrm{B}}^{(i)}\right)} \tag{18}$$

where $\hat{\mathbf{F}}_{\mathrm{A}}^{(i)}$ and $\hat{\mathbf{F}}_{\mathrm{B}}^{(i)}$ are the i-th column vectors ([n×1]) of the flexibility matrices $\mathbf{F}_{\mathrm{A}}$ and $\mathbf{F}_{\mathrm{B}}$, for structures A and B, respectively. $\mathbf{MAC\hat{F}LEX}$ is a vector with as many values as the number of columns in the flexibility matrices. Again, we could consider a full **MACFLEX** matrix by taking different vectors into account, instead of the i-th vector for both structures, but there is no point in that as again the diagonal terms of the **MACFLEX** matrix are the important ones.

By multiplying the n individual values of the $\mathbf{MAC\hat{F}LEX}$ vector, we obtain the *MACFLEX* scalar value as follows:

$$MACFLEX = \prod_{i=1}^{n} \mathbf{MAC\hat{F}LEX}_i \tag{19}$$

In our example, for any number of known modes (1, 2, 3 or 4), it is:

$$\mathbf{MAC\hat{F}LEX}(\mathrm{A},\mathrm{A}) = \mathbf{MAC\hat{F}LEX}(\mathrm{B},\mathrm{B}) = \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix} \tag{20}$$

However, the $\mathbf{MAC\hat{F}LEX}(\mathrm{A},\mathrm{B})$ = $\mathbf{MAC\hat{F}LEX}(\mathrm{B},\mathrm{A})$ vector changes depending on the number of the known eigenmodes, as shown in the table below (shown as the transpose, in column format).

No of known modes	**1**	**2**	**3**	**4**
1st DOF	0.9950	0.9929	0.9932	0.9933
2nd DOF	0.9950	0.9936	0.9939	0.9939
3rd DOF	0.9950	0.9947	0.9948	0.9948
4th DOF	0.9950	0.9957	0.9956	0.9956
5th DOF	0.9950	0.9963	0.9962	0.9962
6th DOF	0.9950	0.9965	0.9965	0.9965
7th DOF	0.9950	0.9967	0.9967	0.9967
8th DOF	0.9950	0.9972	0.9973	0.9973
9th DOF	0.9950	0.9978	0.9980	0.9979
MACFLEX	**0.9557**	**0.9621**	**0.9628**	**0.9629**

Table 6. $\mathbf{MAC\hat{F}LEX}(\mathrm{A},\mathrm{B})^{T}$ and corresponding *MACFLEX* scalar value for 1, 2, 3 or 4 known eigenmodes, for our example.

Figure 4 shows the values of MTMAC and MACFLEX criteria for different number of known eigenmodes (1, 2, 3 or 4), for our example.

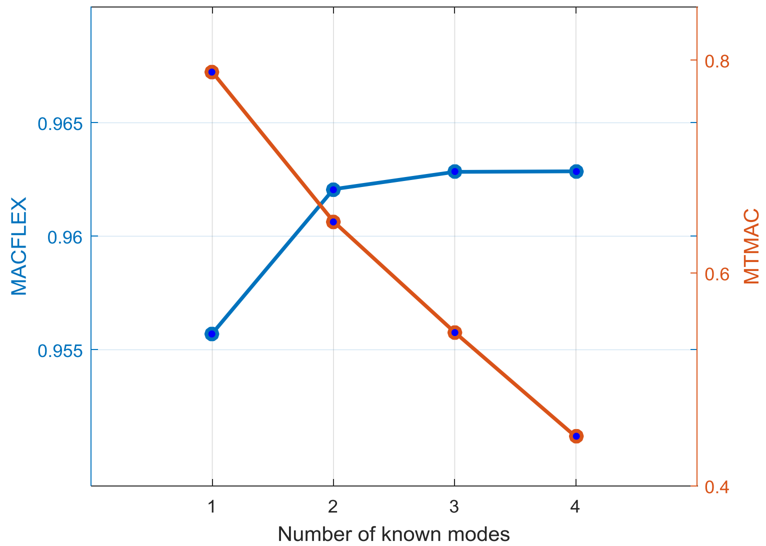


Figure 4. Comparison of *MACFLEX* and *MTMAC* scalar values for 1, 2, 3, and 4 known eigenmodes.

3. DAMAGE IDENTIFICATION AS AN OPTIMIZATION PROBLEM

3.1. Problem formulation

Setting-up an objective function, selecting the updating parameters and using robust optimization algorithms are three crucial steps in structural identification. They require deep physical insight and usually trial-and-error procedures have to be used. In our case, the damage identification problem is considered as an unconstrained optimization problem where the design variables denote the extent of damage of every single element of the structure. In this sense, the number of design variables is equal to the number of elements in the structure. Single beam elements are used to represent the structure of the numerical example. It has been assumed that no alteration occurs before and after damage related to the mass, which is acceptable in most real applications. Therefore, the parameterization of the damage has been represented by a reduction factor or damage index of the element bending stiffness. This damage index, d_e, for a damaged element *e* represents the relative variation of the damaged element bending stiffness, $(EI)_{e,d}$ to the initial (undamaged) bending stiffness $(EI)_e$, as follows:

$$d_e = 1 - \frac{(EI)_{e,d}}{(EI)_e} \tag{21}$$

This definition of a damage index for each element of the structure allows estimating not only the damage extent but also the damage location since the damage identification is carried out at the element level. The damage index can take values between 0 (no damage) and 1 (100% damage, no stiffness), although for numerical stability purposes, the maximum damage has to be limited to a value slightly below 1 (i.e. 0.999) or the structure can become a mechanism that cannot be analyzed and numerical instabilities will occur.

The objective function has to reflect the deviation between the numerical prediction and the real behavior of the structure. For this reason, an objective function may be formulated in terms of the discrepancy between FE and experimental quantities. The following four objective functions have been considered in this study, corresponding to the four different modal correlation criteria (MAC, MTMAC, CoMAC and MACFLEX respectively) between the real damage (according to each examined damage scenario) and the damage which is estimated by the finite element model:

- $F_1 = 1 -$ MAC
- $F_2 = 1 -$ MTMAC
- $F_3 = 1 -$ COMAC
- $F_4 = 1 -$ MACFLEX

The minimum value (target value) for each objective function is zero. At this point it has to be noted that in real life, the dynamics properties (eigenvalues and eigenmodes) of the real damaged structure would have to be determined (measured) by experiment. In our case, these properties are also calculated numerically using a FE “real damage” model, which is perfectly acceptable for the purposes of the present study and does not cause any problems or limitations to the procedure.

3.2. The differential evolution algorithm

Choosing the proper search algorithm for solving an optimization problem is not a straightforward procedure. In the past a number of studies have been published where structural optimization are solved using the metaheuristic search algorithms and especially those based on adopting Darwinian principles of evolutionary process. These algorithms achieve efficient performance for a wide range of combinatorial optimization problems. Among the plethora of such algorithms, the differential evolution (DE) algorithm is adopted in this study to solve the optimization problem of Section 3.1.

Differential evolution (DE) is a stochastic population-based evolutionary algorithm for global optimization, introduced by Storn & Price [17]. It follows the standard evolutionary algorithm flow with some significant differences in mutation and selection process. The simplicity of DE algorithm is based on only three tunable parameters, the mutation factor $F \in [0,2]$, the crossover probability $CR \in [0,1]$ and the total number or particles (population size) *NP*. The fundamental idea behind DE is the use of vector differences by choosing randomly selected vectors, and then taking their difference as a means to perturb the parent vector with a special kind operator and probe the search space. Several variants of DE have been proposed so far [18], but this study is focused on the nominal approach (DE/rand/1/bin). According to this, each of the members of the population undergoes mutation and crossover. Once crossover occurs, the offspring is compared to the parent, and whichever fitness is better moves to the next generation (selection process). In more detail:

We consider an optimization problem with *D* dimensions. First, all individuals $\boldsymbol{x}$ are initialized at random positions in the search-space. After initialization each member of the population $\boldsymbol{x}$ undergoes mutation and a donor vector $\boldsymbol{v}$ is generated such as:

$$\boldsymbol{v} = \boldsymbol{a} + F \cdot (\boldsymbol{b} - \boldsymbol{c}) \tag{22}$$

where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are three individuals from the population at random, which must be distinct from each other as well as from individual $\boldsymbol{x}$ ($\boldsymbol{x} \neq \boldsymbol{a} \neq \boldsymbol{b} \neq \boldsymbol{c}$).

In the next step the crossover operator is applied by generating the trial vector $\boldsymbol{u}$ which is defined either from the *i*-th component (v_i) of $\boldsymbol{v}$ or the *i*-th component (x_i) of $\boldsymbol{x}$, with probability *CR* as follows:

$$u_i = \begin{cases} v_i & \text{if } r_i \le CR \text{ or } i = R \\ x_i & \text{otherwise} \end{cases} \qquad i = \{1, 2, ..., D\} \tag{23}$$

where r_i is a random number with uniform distribution, $r_i \in U[0,1]$, and *R* is a random integer in [1, 2, …, *D*] which ensures that in any case, after the crossover operation it is $\boldsymbol{u} \neq \boldsymbol{x}$. The last step of the generation procedure is the implementation of the selection operator where the target vector $\boldsymbol{x}$ is compared to the trial vector $\boldsymbol{u}$. If the trial vector $\boldsymbol{u}$ has a better fitness value, then the individual $\boldsymbol{x}$ is replaced in the population with the trial vector $\boldsymbol{u}$ as follows:

$$\boldsymbol{x}' = \begin{cases} \boldsymbol{u} & \text{if } f(\boldsymbol{u}) < f(\boldsymbol{x}) \\ \boldsymbol{x} & \text{otherwise} \end{cases} \tag{24}$$

where *f* is the objective function to be minimized and $\boldsymbol{x}'$ is the new design vector for the next generation. The optimization procedure finished when the maximum number of generations has been reached.

4. NUMERICAL EXAMPLES

A simply supported beam [19] is analyzed in this section to illustrate the performance of the proposed methodology and the different criteria. The geometry, boundary conditions and finite element mesh of the beam are shown in Figure 5. The beam has a total length of 6 m and it is discretized by 10 equal length beam elements of rectangular cross section b (width) x h (height)

= 0.25 m × 0.20 m. The beam is considered to have a Young's modulus E equal to 30 GPa and a density ρ equal to 2500 kg/m^3.

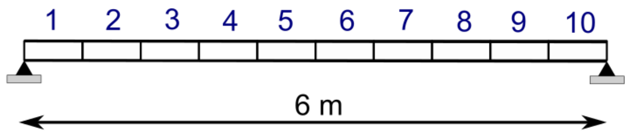
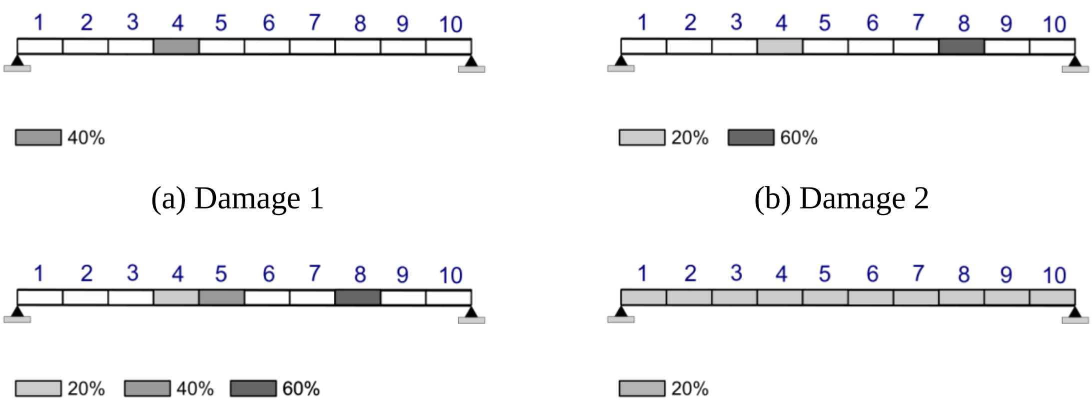


Figure 5. The beam structure under investigation.

The parameters of the DE optimization problem are the following:

- D = 10 (dimension of the problem)
- NP = 40 (population size)
- F = 0.6 (mutation factor)
- CR = 0.9 (crossover probability)
- MAXGEN = 3000 (maximum numbers of generations)

Four different damage scenarios are considered: (1) A single-element damage scenario (Figure 6a); (2) a two-element damage case (Figure 6b); (3) a three-element damage case (Figure 6c); and (4) a uniform damage case (Figure 6d). The finite element model of the beam is based on Euler–Bernoulli assumption of the planar elements with two degrees of freedom per node (the axial deformation is ignored).



(a) Damage 1

(b) Damage 2

(c) Damage 3

(d) Damage 4

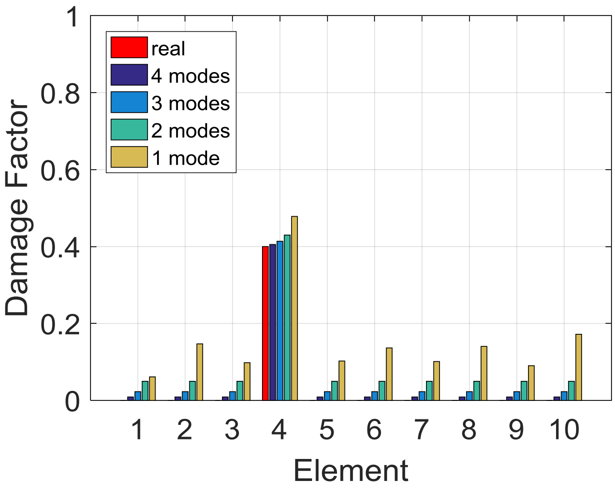
Figure 6. The four different damage scenarios.

4.1. Results

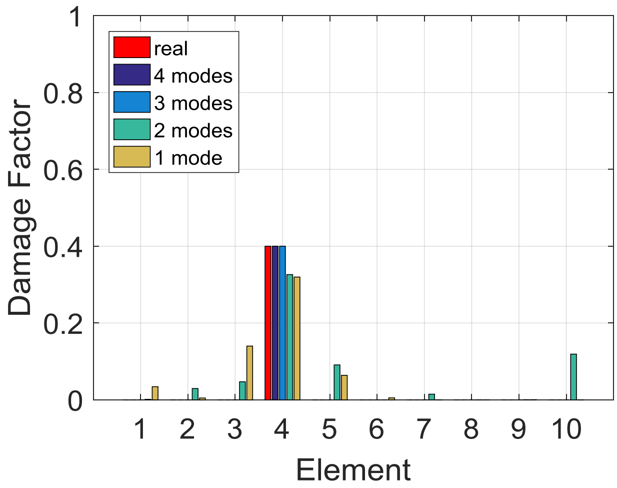
The same optimization algorithm has been applied to all damage scenarios. For each damage scenario, the four different modal correlation criteria have been used for the formulation of the objective function. For each criterion, the number of known eigenmodes varies from 1 to 4. The results are presented in bar charts, where the target damage (real damage) is always denoted in red color and the other colored bars denoted the damage estimation by the optimization procedure.

4.1.1. Damage 1 scenario (single-element damage)

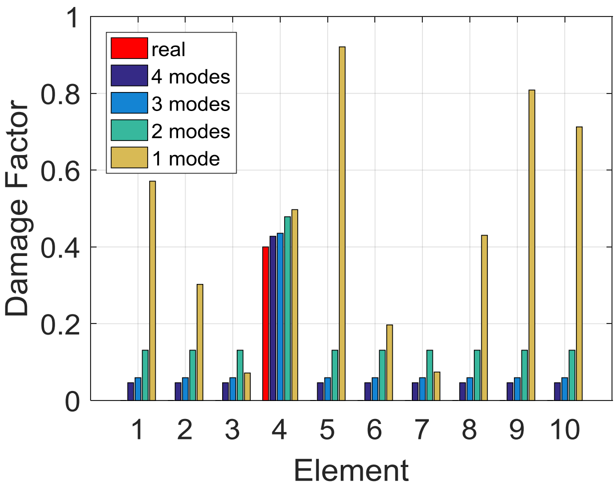
Figure 7 shows the performance of the four different criteria for the first damage scenario. We see that the MTMAC criterion shows very good performance, since it manages to identify the damage almost 100% in the cases where 3 or 4 eigenmodes are known, while a good performance is also recorded for the difficult cases of 2 or even only 1 known eigenmode. The MAC criterion shows also good performance, but again it cannot be compared to the performance of the MTMAC criterion.



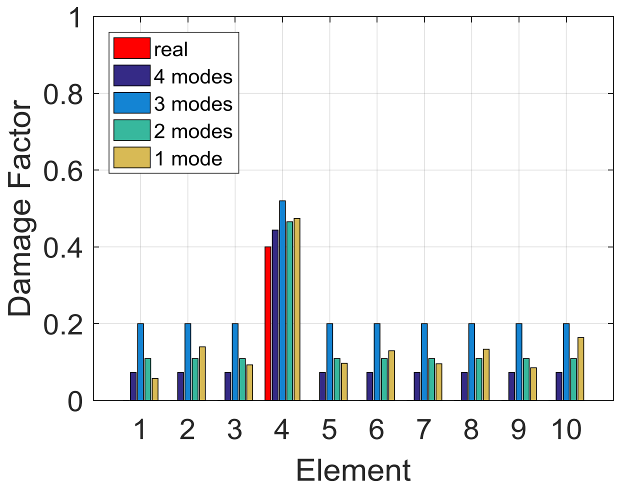
(a) MAC



(b) MTMAC



(c) CoMAC

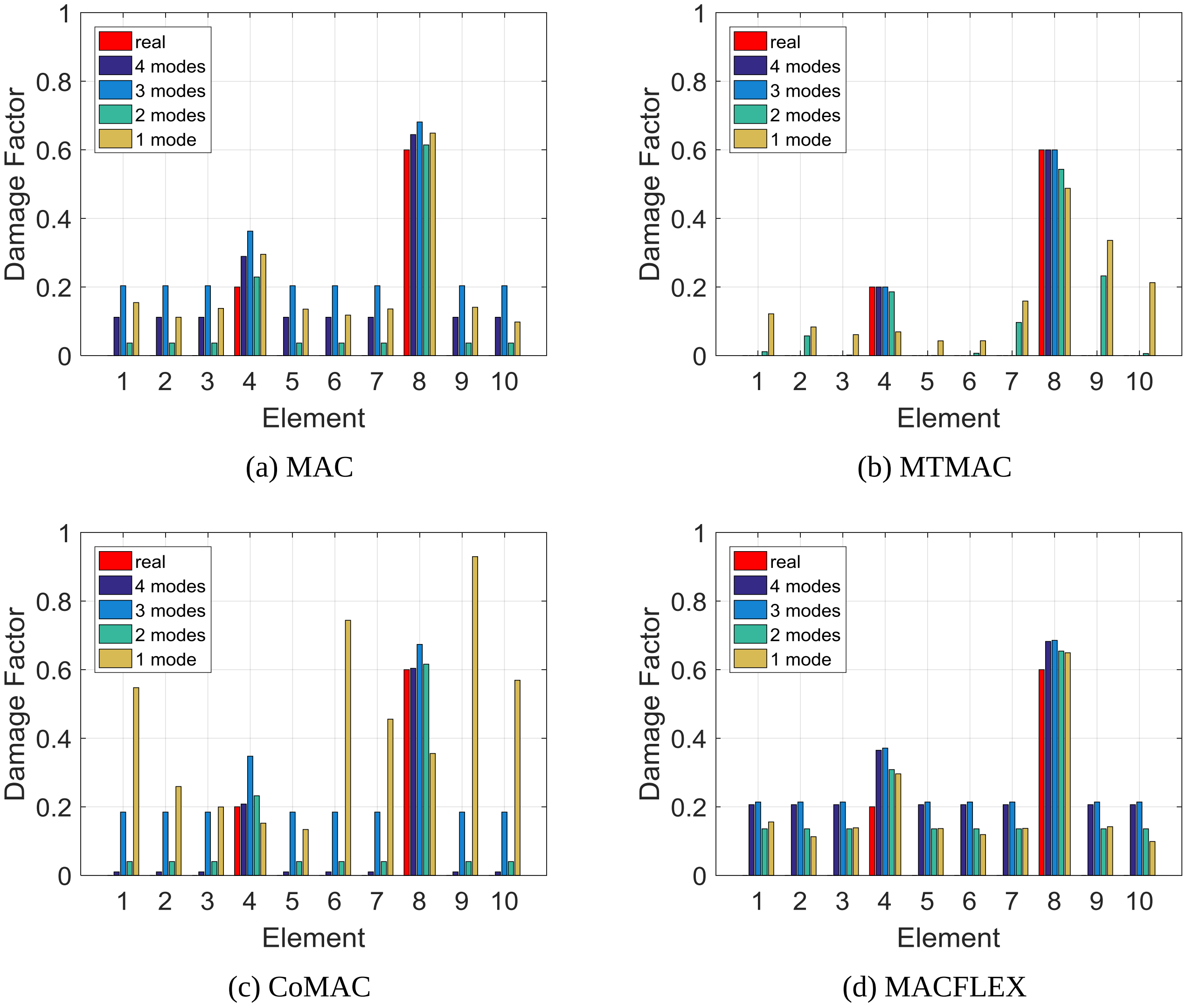


(d) MACFLEX

Figure 7. Performance of the four different criteria for the single-element damage scenario (Damage 1).

4.1.2. Damage 2 scenario (two-element damage)

Figure 8 shows the performance of the four different criteria for the second damage scenario. This damage scenario appears to be more difficult than the first one. Again, we see that the MTMAC criterion shows exceptional performance in the cases where 3 or 4 eigenmodes are known. The other criteria appear not to exhibit a very good performance, even in the cases where 4 eigenmodes are known.



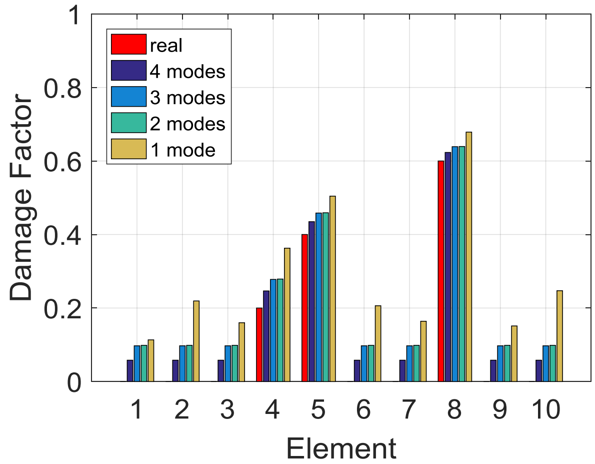
(a) MAC (b) MTMAC

(c) CoMAC (d) MACFLEX

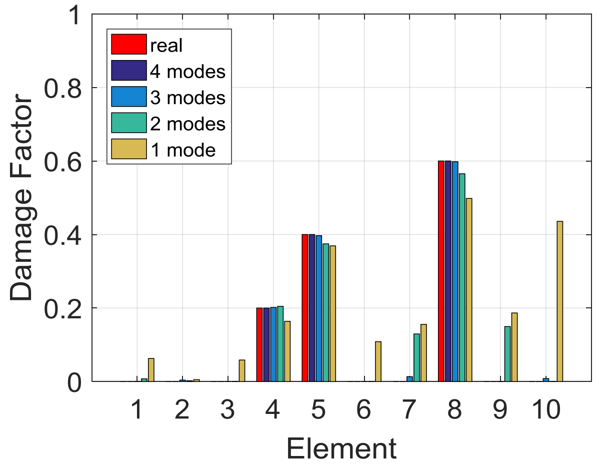
Figure 8. Performance of the four different criteria for the two-element damage scenario (Damage 2).

4.1.3. Damage 3 scenario (three-element damage)

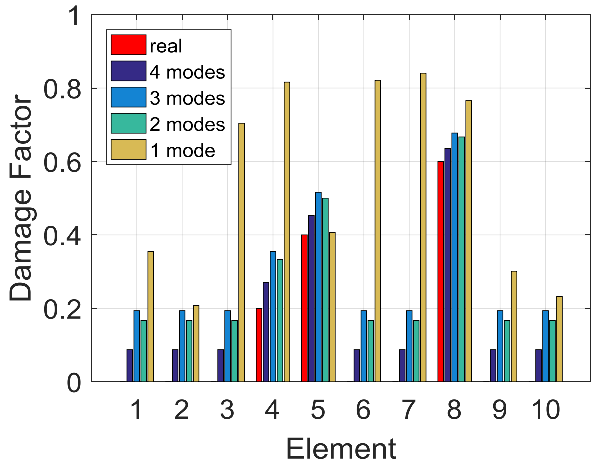
Figure 9 shows the performance of the four different criteria for the third damage scenario. Again the trend is the same. Only the MTMAC shows excellent performance, especially in the cases of 3 or 4 known eigenmodes, while the other criteria fail to identify the location and extent of damage adequately, even in the cases where 4 eigenmodes are known.



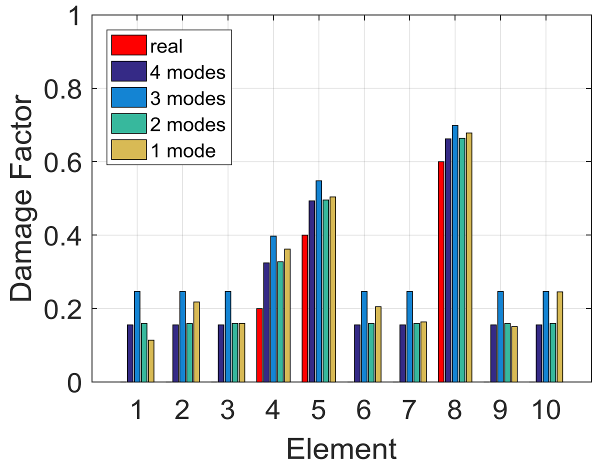
(a) MAC



(b) MTMAC



(c) CoMAC

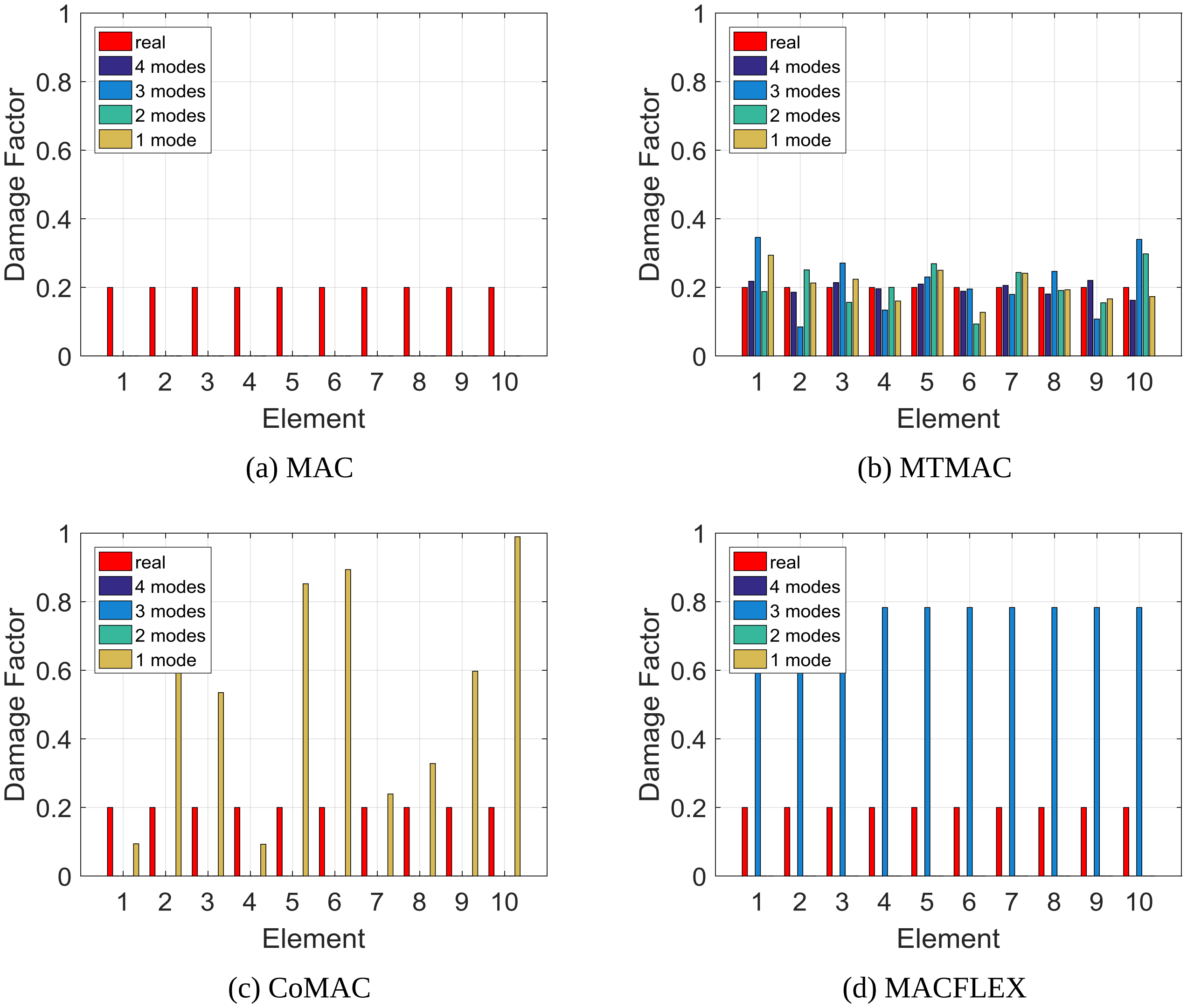


(d) MACFLEX

Figure 9. Performance of the four different criteria for the three-element damage scenario (Damage 3).

4.1.4. Damage 4 scenario (uniform damage)

Figure 10 shows the performance of the four different criteria for the last damage scenario. The uniform damage appears to be the most difficult scenario. With the exception of MTMAC, the three other criteria completely fail to identify the location or extent of damage and they seem to just not be working at all. On the other hand, MTMAC shows very good performance when 4 eigenmodes are known, while its performance in the case where 3 eigenmodes are known can be still considered as acceptable. The overall performance of MTMAC is good but it is not as good as in the three other damage scenarios.



(a) MAC (b) MTMAC

(c) CoMAC (d) MACFLEX

Figure 10. Performance of the four different criteria for the uniform damage scenario (Damage 4).

The reason that the other three criteria (MAC, CoMAC and MACFLEX) fail completely is that only the MTMAC criterion contains also information about the eigenvalues (or eigenperiods) of the structure. The other criteria take into account only information about the eigenmodes. It is known that in the special case of uniform damage, the eigenmodes of the structure themselves do not change and the only property that changes is the eigenperiod which becomes larger (the structure becomes more flexible). As a result, only the MTMAC criterion manages to identify this special kind of damage and can be trustworthy for such damage cases.

5. CONCLUSIONS

- In this paper, the performance of different modal correlation criteria in structural damage identification was investigated. The structural damage identification problem was treated as an optimization problem which was solved using the differential evolution optimization algorithm.
- The DE algorithm proved to be very efficient and robust in all cases, while the four correlation criteria exhibited different performances for each damage case.
- In general, the MTMAC criterion showed excellent performance, managing to identify almost 100% the location and extent of damage for all damage cases, when 3 or 4 eigenmodes were known. In the cases were limited data were available (1 or 2 known eigenmodes), this criterion showed also an acceptable performance which was the best among the different criteria.
- The other criteria showed good performance only in some individual damage cases, but their general performance was not reliable, especially when a smaller number of eigenmodes were considered (1, 2 or 3).
- Some damage scenarios were more difficult than others. The most difficult was the uniform damage (4^{th}) scenario. Only the MTMAC criterion managed to give a good estimation for this damage case and again, the quality of the solution was not perfect, even in the case of 4 known eigenmodes.

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