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DESIGN OF RC SECTIONS IN THE ULTIMATE LIMIT STATE UNDER BENDING AND AXIAL FORCE ACCORDING TO EC2

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Abstract. *In the imminent future the design of concrete structures in Europe will be governed by the application of Eurocode 2 (EC2). In particular, EC2 – Part 1-1 [1] deals with the general rules and rules for concrete buildings. An important aspect of the design is specifying the necessary tensile (and compressive, if needed) steel reinforcement required for a Reinforced Concrete (RC) section, in order to ensure that the RC member will be able to resist the design loads.*

According to EC2-Part 1-1 three different simplified diagrams for the stress-strain behavior of concrete for RC sections design can be assumed: (a) the equivalent rectangular stress block, (b) the parabolic-rectangular stress-strain relation and (c) the bi-linear stress-strain relation as a simplification of the parabolic-rectangular case. In this study the three approaches suggested by EC2-Part 1-1 are investigated for the design of rectangular RC cross-sections with tensile steel reinforcement to resist loading due to bending moment and axial force. The tensile strength of concrete is neglected and concrete is supposed to work only in compression. For each case analytical relations are extracted in detail with a step-by-step detailed procedure, the relevant assumptions are highlighted and results for four beam design examples are finally presented.

1 LITERATURE REVIEW

Rosca and Petru [2] study the design of a reinforced concrete section subjected to bending using the two stress–strain relationships mentioned in EC2, namely the parabola-rectangle stress distribution and the rectangular distribution, and the differences are underlined. Two dimensionless quantities are used to convert the parabola-rectangle stress distribution to an equivalent concentrated force for the concrete in compression. Also analytical relations which determine the limit between single reinforcement (only tensile) and double reinforcement (tensile and compressive) are provided. The results drawn from the use of these two stress distributions, namely, parabola–rectangle and rectangle, showed that the differences between the amounts of reinforcement are less than 1% for singly reinforced sections and less than 2% for doubly reinforced sections.

Dulinskas and Zabulionis in [3] and [4] propose a method for the substitution of the non-linear stress diagram with descending branch with an equivalent rectangular stress block when the non-linear stress-strain relationship for concrete in compression is described according to EC2. Analytical relationships in explicit form for area, the first moment of area, the coordinate of centroid of the nonlinear stress diagram with descending branch, the ratio between the depth of the rectangular stress block and that of the equivalent nonlinear stress diagram with descending branch in respect to the concrete strength are given. Coefficients suitable for the substitution of parabola stress diagram with descending branch given in EC2 with an equivalent rectangular stress block are presented. These coefficients have to ensure that the substitution is equivalent, i.e. the carrying capacity of the compression zone calculated using either of the two stress diagrams should be the same.

Židonis in [5] tries to replace the nonlinear stress-strain diagram of concrete adopted by EC2 for structural analysis by another more general curvilinear diagram which relates stress and strain of concrete. The new stress-strain diagram permits direct integration without the need to discretize the stress-strain curve. Thus it makes the integration easier and can be applied to the concrete classes from C8/10 up to C90/105. Analytical stress-strain relations are presented for concrete which can fit the stress-strain curves specified in EC2 within an error of 1.5%. Finally, examples of application of the proposed stress-strain diagram are illustrated.

In [6] a method is presented and formulas are provided for application of non-linear concrete stress diagram for cross-section strength calculation in accordance with the limit state (partial factors) method. Commonly reinforced concrete flexural members with rectangular compression zone and the neutral axis within the cross-section are considered (beam-type members); the effect of the descendant part of stress-strain diagram on strength of cross-sections of beam type members is investigated and the limit between commonly and abundantly reinforced concrete beams is determined. Finally the results of the new method are compared with those of EC2, where rectangular compression zone stress diagram for concrete is assumed. A table is extracted in which all necessary information needed to perform design for bending of a reinforced concrete section for all concrete strength classes are shown.

Although the above studies deal with the application of the most suitable stress-strain diagram for concrete for the “optimal” design of cross sections using different approaches, to the authors’ knowledge, there is no study in which explicit formulas and/or graphs are provided to achieve the design of RC concrete sections according to EC2-Part 1-1 for all the three stress-strain relations provided. In the present study, the three suggested stress-strain diagrams of EC2 are investigated and analytical formulas are given for the step-by-step design or RC sections according to any one of the three design approaches.

2 DESIGN ASSUMPTIONS

The following design assumptions are made:

- Design is based on characteristic concrete cylinder strengths, not cube strengths.
- Plane sections remain plane.
- Strain in the bonded reinforcement, whether in tension or compression, is the same as that in the surrounding concrete.
- Tensile strength of the concrete is ignored.
- Concrete stress distribution is considered according to the three cases of Eurocode 2, as will be shown in detail in the next paragraphs.
- Stress in steel reinforcement is considered according to the stress-strain relation of Eurocode 2 for steel, as will be shown in detail in the next paragraphs.

3 CONCRETE PROPERTIES

According to EC2-1-1 the compressive strength of concrete is denoted by concrete strength classes which relate to the characteristic (5%) cylinder strength f_{ck}, or cube strength $f_{ck,cube}$, in accordance with EN 206-1. Higher strengths of concrete are covered by Eurocode 2, up to class C90/105. The strength classes for concrete are presented in the table below.

f_{ck} (MPa)	12	16	20	25	30	35	40	45	50	55	60	70	80	90
$f_{ck,cube}$ (MPa)	15	20	25	30	37	45	50	55	60	67	75	85	95	105

Table 1. Strength classes for concrete.

where f_{ck} is the characteristic compressive cylinder strength of concrete at 28 days and $f_{ck,cube}$, is the corresponding cube strength. The value of the design compressive strength is defined as

$$f_{cd} = a_{cc} \frac{f_{ck}}{\gamma_c} \tag{1}$$

where:

- γ_c is the partial safety factor for concrete at the Ultimate Limit State, which is given in Table 2.1N of EC2-1-1. For persistent and transient design situations, γ_c=1.5
- a_{cc} is the coefficient taking account of long term effects on the compressive strength and of unfavourable effects resulting from the way the load is applied. The value of a_{cc} for use in a country should lie between 0.8 and 1.0 and may be found in its National Annex. The recommended value is 1.

It should be noted that higher concrete strength shows more brittle behavior, reflected by shorter horizontal branch, as will be shown in the stress-strain relations, later.

4 CONCRETE STRESS-STRAIN RELATIONS FOR THE DESIGN OF CROSS-SECTIONS

Eurocode 2 Part 1-1 suggests the use of three approaches for the stress-strain relations of concrete for the design of cross sections:

- Parabola-rectangle diagram (more detailed)
- Bi-linear stress-strain relation (less detailed)
- Rectangular stress distribution (simplest)

The three above approaches will be described in detail in the following sections.

4.1 Parabola-rectangle diagram for concrete under compression

According to EC2-1-1 (Paragraph 3.1.7), for the design of cross-sections, the following stress-strain relationship may be used:

$$\sigma_c = \begin{cases} f_{cd}\left[1-\left(1-\dfrac{\varepsilon_c}{\varepsilon_{c2}}\right)^n\right] & \text{for} \quad 0 \le \varepsilon_c < \varepsilon_{c2} \\ f_{cd} & \text{for} \quad \varepsilon_{c2} \le \varepsilon_c \le \varepsilon_{cu2} \end{cases} \tag{2}$$

where:

- n is the exponent given by

$$n = \begin{cases} 2.0 & \text{for} \quad f_{ck} \le 50MPa \\ 1.4+23.4\left(\dfrac{90-f_{ck}}{100}\right)^4 & \text{for} \quad 50 < f_{ck} \le 90MPa \end{cases} \tag{3}$$

- ε_{c2} is the strain at reaching the maximum strength given by

$$\varepsilon_{c2}(‰) = \begin{cases} 2.0 & \text{for} \quad f_{ck} \le 50MPa \\ 2.0+0.085\left(f_{ck}-50\right)^{0.53} & \text{for} \quad 50 < f_{ck} \le 90MPa \end{cases} \tag{4}$$

- ε_{cu2} is the ultimate strain given by

$$\varepsilon_{cu2}(‰) = \begin{cases} 3.5 & \text{for} \quad f_{ck} \le 50MPa \\ 2.6+35\left(\dfrac{90-f_{ck}}{100}\right)^4 & \text{for} \quad 50 < f_{ck} \le 90MPa \end{cases} \tag{5}$$

The above equation is depicted in the following figure, where compressive stresses (and strains) are shown as positive.

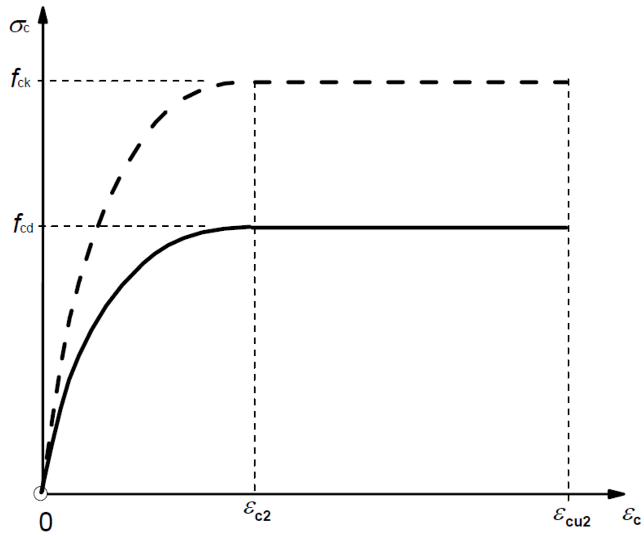

Figure 1: Parabola-rectangle diagram for concrete under compression.

4.2 Bi-linear stress-strain relation

According to Paragraph 3.1.7(2) of EC2-1-1, other simplified stress-strain relationships may be used if equivalent or more conservative than the one defined in Eq. (2), for instance bi-linear according to the following figure, where again compressive stress and shortening strain are shown as absolute values.

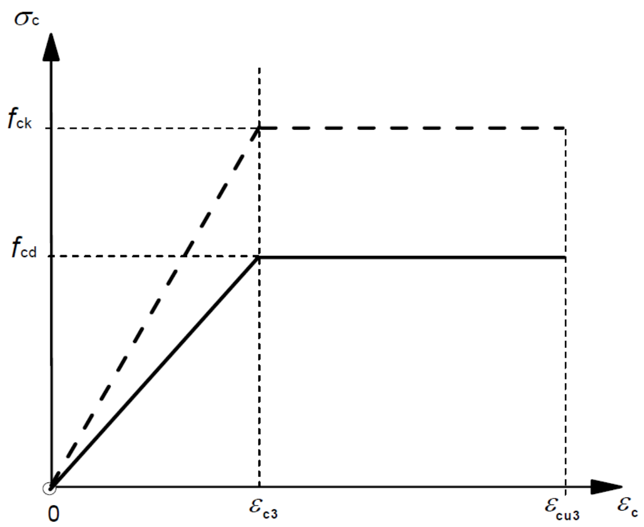

Figure 2: Bi-linear stress-strain relation.

where the values of ε_{c3} and ε_{cu3} are given by

$$\varepsilon_{c3}(‰)=\begin{cases} 1.75 & \text{for} \quad f_{ck}\leq 50MPa \\ 1.75+0.55\cdot\dfrac{f_{ck}-50}{40} & \text{for} \quad 50< f_{ck}\leq 90MPa \end{cases} \tag{6}$$

$$\varepsilon_{cu3}(‰) = \begin{cases} 3.5 & \text{for} \quad f_{ck} \le 50MPa \\ 2.6 + 35\left(\dfrac{90 - f_{ck}}{100}\right)^4 & \text{for} \quad 50 < f_{ck} \le 90MPa \end{cases} \tag{7}$$

4.3 Rectangular stress distribution

According to Paragraph 3.1.7(3) of EC2-1-1, a rectangular stress distribution may be assumed for concrete, as shown in the following figure.

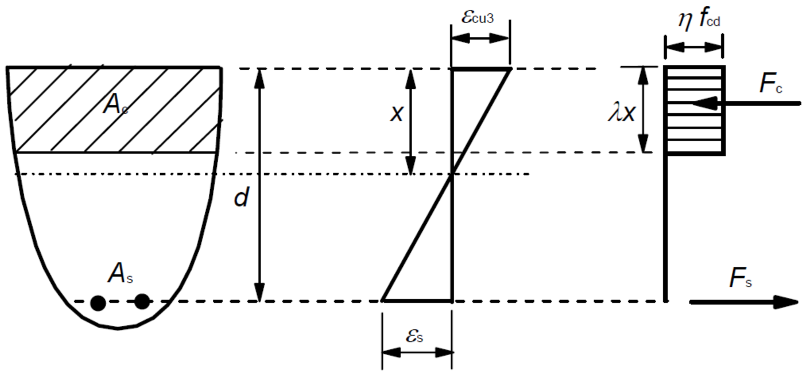

Figure 3: Rectangular stress distribution.

where:

- d is the effective depth of a cross-section
- x is the neutral axis depth
- A_s is the cross sectional area of the tensile steel reinforcement
- F_c is the concrete force (compressive, positive, as in the figure)
- F_s is the steel reinforcement force (tensile, positive, as in the figure)
- The factor λ defining the effective height of the compression zone and the factor η defining the effective strength, are calculated from:

$$\lambda = \begin{cases} 0.8 & \text{for} \quad f_{ck} \le 50MPa \\ 0.8 - \dfrac{f_{ck} - 50}{400} & \text{for} \quad 50 < f_{ck} \le 90MPa \end{cases} \tag{8}$$

$$\eta = \begin{cases} 1.0 & \text{for} \quad f_{ck} \le 50MPa \\ 1.0 - \dfrac{f_{ck} - 50}{200} & \text{for} \quad 50 < f_{ck} \le 90MPa \end{cases} \tag{9}$$

Note: If the width of the compression zone decreases in the direction of the extreme compression fibre, the value $\eta \cdot f_{cd}$ should be reduced by 10%.

5 STEEL PROPERTIES

The design strength for steel is given by

$$f_{yd} = \frac{f_{yk}}{\gamma_s} \tag{10}$$

where:

- γ_s is the partial safety factor for steel at the Ultimate Limit State, which is given in Table 2.1N of EC2-1-1. For persistent and transient design situations, γ_s=1.15

- f_{yk} is the characteristic yield strength of steel reinforcement.

Table C.1 of Annex C of EC2-1-1 gives the properties of reinforcement suitable for use with the Eurocode. The properties are valid for temperatures between -40ºC and 100ºC for the reinforcement in the finished structure. Any bending and welding of reinforcement carried out on site should be further restricted to the temperature range as permitted by EN 13670.

Product form	Bars and de-coiled rods			Wire Fabrics			Requirement or quantile value (%)
Class	A	B	C	A	B	C	-
Characteristic yield strength f_{yk} or $f_{0,2k}$ (MPa)	400 to 600						5,0
Minimum value of $k = (f_t/f_y)_k$	≥1,05	≥1,08	≥1,15 <1,35	≥1,05	≥1,08	≥1,15 <1,35	10,0
Characteristic strain at maximum force, ε_{uk} (%)	≥2,5	≥5,0	≥7,5	≥2,5	≥5,0	≥7,5	10,0
Bendability	Bend/Rebend test			-			
Shear strength	-			0,3 A f_{yk} (A is area of wire)			Minimum
Maximum deviation from nominal mass (individual bar or wire) (%) — Nominal bar size (mm) ≤ 8 > 8	 ± 6,0 ± 4,5						5,0

Table 2. Properties of steel reinforcement (Table C.1 of Annex C of EC2-1-1).

The application rules for design and detailing in Eurocode 2 are valid for a specified yield strength range, f_{yk} from 400 to 600 MPa. The yield strength f_{yk} is defined as the characteristic value of the yield load divided by the nominal cross sectional area. The reinforcement shall have adequate ductility as defined by the ratio of tensile strength to the yield stress, $(f_t/f_y)_k$ and the elongation at maximum force, ε_{uk}.

5.1 Steel stress-strain relations for the design of cross-sections

For normal design, either of the following assumptions may be made for the stress-strain relation for steel (see figure below):

1. An inclined top branch with a strain limit of ε_{ud} and a maximum stress of $k{\cdot}f_{yk}/\gamma_s$ at ε_{uk}, where $k=(f_t/f_y)_k$ (see the above Table 2).
2. A horizontal top branch without the need to check the strain limit.

The value of ε_{ud} for use in a country may be found in its National Annex. The recommended value is $0.9\varepsilon_{uk}$. The design value of the steel modulus of elasticity E_s may be assumed to be 200 GPa.

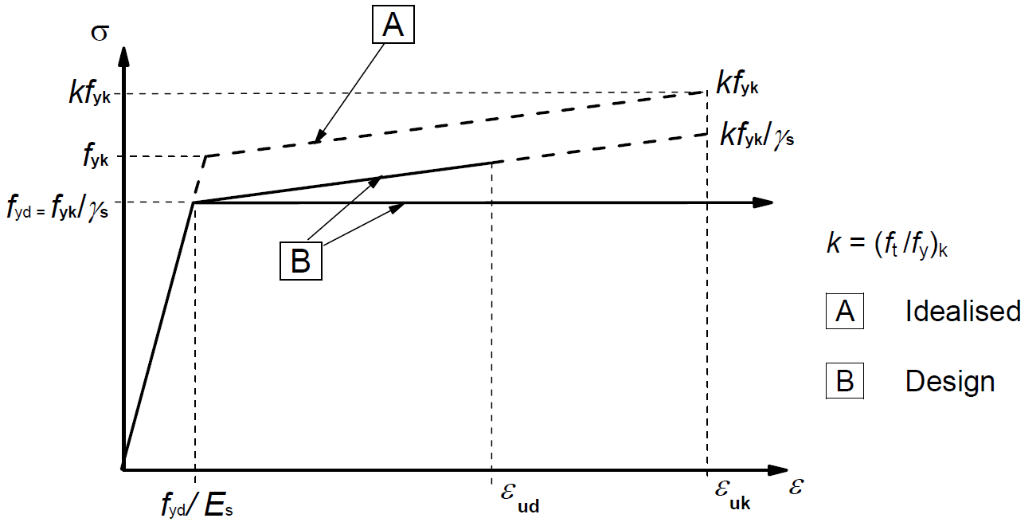

Figure 4: Idealised and design stress-strain diagrams for reinforcing steel (for tension and compression)

In the present study we will use the second approach, assuming a horizontal top branch for the steel stress-strain relation, but also limiting the maximum strain to ε_{ud}, as shown in the following figure.

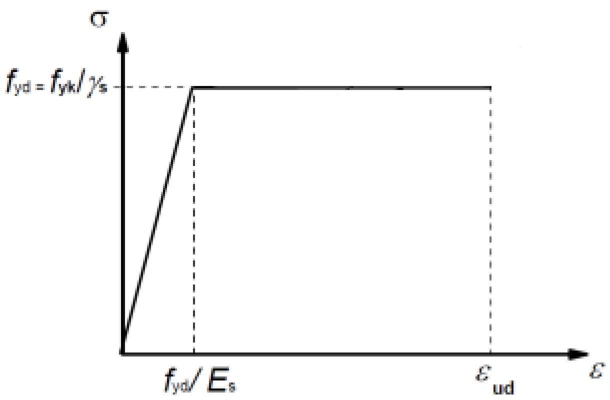

Figure 5: Design stress-strain diagram for reinforcing steel (for tension and compression) used in the present study.

6 INVESTIGATION OF THE SIMPLIFIED RECTANGULAR STRESS DISTRIBUTION CASE

The figure below shows a typical rectangular cross section and the distribution of strains and stresses (forces).

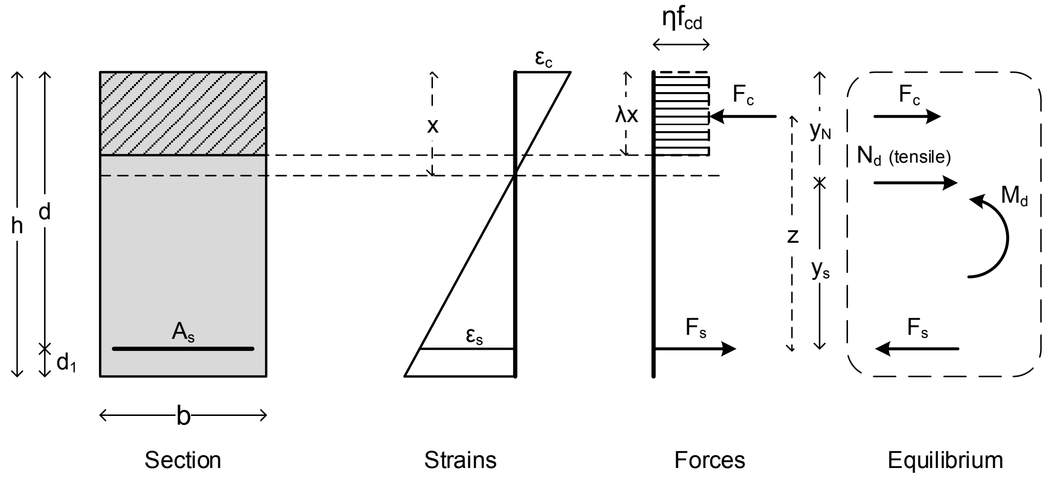

Figure 6: Cross section, strain and forces distribution and section equilibrium.

In the above figure:

- h is the height of the rectangular section
- d_1 is the distance from the lower edge of the section to the center of the reinforcement
- ε_s is the tensile strain in the steel reinforcement
- ε_c is the compressive strain in the concrete upper edge
- λ is a factor defining the effective height of the compression zone
- η is a factor defining the effective strength of the compression zone
- M_d is the applied external bending moment (puts the lower edge of the section in tension if positive)
- N_d is the applied external axial force (tensile for the section if positive), applied at a position y_N measured from the top of the section towards the lower edge of it. Note: If we have central tension, then $y_n=h/2$
- y_s is the distance from the steel reinforcement to the position of the external applied axial force
- z is the distance of the concrete force F_c from the steel reinforcement.

The goal of the design is to calculate the needed cross sectional area of steel reinforcement A_s. In order to calculate A_s, we need first to calculate the unknown quantities x and z. We move the external force N_d to the position of the steel reinforcement and we have the figure below.

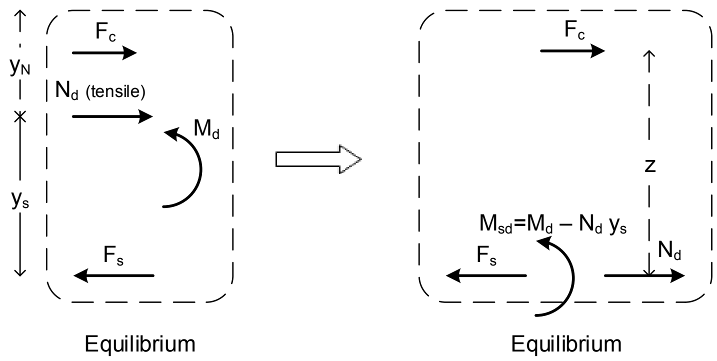

Figure 7: Equilibrium after moving the external force N_d to the position of the steel reinforcement.

From the equilibrium of the section in the x-direction, we have:

$$\Sigma F_x = 0 \Rightarrow F_c + N_d - F_s = 0 \Rightarrow F_s = F_c + N_d \tag{11}$$

We have also:

$$y_s + y_n = d \Rightarrow y_s = d - y_n \tag{12}$$

The effective bending moment is:

$$M_{sd} = M_d - N_d \cdot y_s \tag{13}$$

From the geometry of the section, we have:

$$z + \frac{\lambda x}{2} = d \Rightarrow z = d - \frac{\lambda x}{2} \tag{14}$$

The concrete force is given by:

$$F_c = \lambda x \cdot nf_{cd} \cdot b \tag{15}$$

From the equilibrium of moments at the position of the steel reinforcement (Figure 7) we have (clockwise moment taken as positive):

$$\Sigma M_{steel} = 0 \Rightarrow F_c \cdot z - M_{sd} = 0 \Rightarrow M_{sd} = F_c \cdot z \tag{16}$$

By substituting Eq. (15) in Eq. (16), we have:

$$M_{sd} = \lambda x \cdot nf_{cd} \cdot b \cdot z \tag{17}$$

By substituting Eq. (14) in Eq.(17), we have:

$$M_{sd} = \lambda x \cdot nf_{cd} \cdot b \cdot \left(d - \frac{\lambda x}{2} \right) = x \cdot \left(\lambda \cdot nf_{cd} \cdot b \cdot d \right) - x^2 \cdot \left(nf_{cd} \cdot b \cdot \frac{\lambda^2}{2} \right) \Rightarrow \tag{18}$$

$$\left(\frac{nf_{cd} \cdot b \cdot \lambda^2}{2} \right) \cdot x^2 - \left(\lambda \cdot nf_{cd} \cdot b \cdot d \right) \cdot x + M_{sd} = 0 \tag{19}$$

The above equation can be written as:

$$Ax^2 + Bx + C = 0 \tag{20}$$

where $$A = \frac{nf_{cd} \cdot b \cdot \lambda^2}{2}, \quad B = -\left(\lambda \cdot nf_{cd} \cdot b \cdot d\right), \quad C = M_{sd} \tag{21}$$

The above quantities A, B and C are all known, so by solving the quadratic Eq. (20) we can determine the quantity *x*. The discriminant of the quadratic equation is:

$$\Delta = B^2 - 4A \cdot C = \left(\lambda \cdot nf_{cd} \cdot b \cdot d\right)^2 - 2nf_{cd} \cdot b \cdot \lambda^2 \cdot M_{sd} \tag{22}$$

The solution of the quadratic equation is:

$$x_{1,2} = \frac{-B \pm \sqrt{\Delta}}{2A} = \frac{d}{\lambda} \pm \frac{\sqrt{\Delta}}{2A} \Rightarrow \begin{cases} x_1 = \dfrac{d}{\lambda} - \dfrac{\sqrt{\Delta}}{2A} \\ x_2 = \dfrac{d}{\lambda} + \dfrac{\sqrt{\Delta}}{2A} \end{cases} \tag{23}$$

Given the requirement that $0 \le x \le d$ and the fact that $\lambda=0.80$ for $f_{ck} \le 50$ MPa and $\lambda<0.80$ for $50<f_{ck} \le 90$ MPa, it is obvious that $d/\lambda>d$, and as a result $x_2>d$ which is not acceptable. So the only acceptable solution is $x=x_1$ and thus:

$$x = \frac{d}{\lambda} - \frac{\sqrt{\Delta}}{2A} \tag{24}$$

After calculating *x*, it is easy to calculate also *z* with Eq. (14), F_c with Eq. (15) and F_s with Eq. (11). We have also:

$$F_s = A_s \cdot \sigma_s \Rightarrow A_s = \frac{F_s}{\sigma_s} \tag{25}$$

In the above equations, σ_s is the steel stress at the Ultimate Limit State (ULS) of the section. The yield strain ε_{ys} for steel is:

$$\varepsilon_{ys} = \frac{f_{yd}}{E_s} \tag{26}$$

where

$$f_{yd} = \frac{f_{yk}}{\gamma_s} \tag{27}$$

if $\varepsilon_s \ge \varepsilon_{ys}$ then the steel works in full stress and $\sigma_s=f_{yd}$. Otherwise, if $\varepsilon_s<\varepsilon_{ys}$ then the steel does not work in full stress and $\sigma_s<f_{yd}$. In general, the steel stress σ_s is given by:

$$\sigma_s = \begin{cases} f_{yd} \cdot \dfrac{\varepsilon_s}{\varepsilon_{ys}} = E_s \cdot \varepsilon_s & \text{if } 0 < \varepsilon_s < \varepsilon_{ys} \\ f_{yd} & \text{if } \varepsilon_s \ge \varepsilon_{ys} \end{cases} \tag{28}$$

In order to determine the area of steel reinforcement A_s, we need to determine the steel stress σ_s and thus we should determine the steel strain ε_s. In order to determine ε_s given the value of x, we first need to know if the steel reinforcement or the concrete is critical (at the ultimate strain) at the Ultimate Limit State of the section.

In the present study, the steel reinforcement is limited to ε_{ud}. If the steel reinforcement is critical, then $\varepsilon_s=\varepsilon_{ud}$ and $\varepsilon_c<\varepsilon_{cu3}$. Otherwise, if the concrete is critical, then $\varepsilon_c=\varepsilon_{cu3}$ and $\varepsilon_s<\varepsilon_{ud}$. In the special case where both materials are critical, then $\varepsilon_s=\varepsilon_{ud}$ and $\varepsilon_c=\varepsilon_{cu3}$. These three possible states are presented in detail in the figure below.

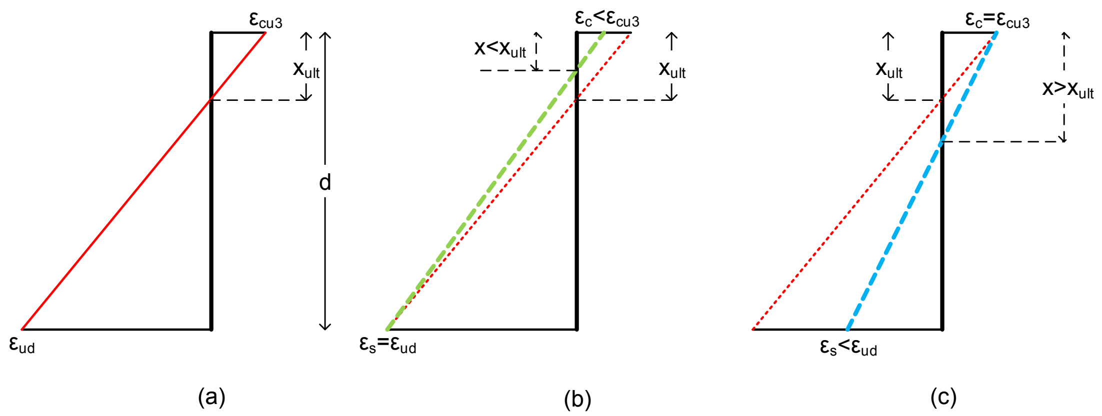

Figure 8: Three possible states of the strains of the cross section.
(a) Both materials are at the ultimate strain, (b) Steel at the ultimate strain, (c) Concrete at the ultimate strain.

In order to find out if the concrete or the steel is critical, we first calculate x_{ult} which is the neutral axis depth for the special case of both materials being critical, as in the figure above – Case (a). It should be noted that this is only a theoretical case and it does not correspond to equilibrium of the cross section. Using similar triangles, we have:

$$\frac{\varepsilon_{cu3}}{x_{ult}}=\frac{\varepsilon_{cu3}+\varepsilon_{ud}}{d}\Rightarrow x_{ult}=\frac{\varepsilon_{cu3}}{\varepsilon_{cu3}+\varepsilon_{ud}}\cdot d \tag{29}$$

Then we have 2 cases:

Case 1: $x < x_{ult}$, as shown in Figure 8 (b)

The steel reinforcement is at the critical strain, $\varepsilon_s=\varepsilon_{ud}$, while $\varepsilon_c<\varepsilon_{cu3}$. In this case, it is ensured that the steel works in full stress, and thus

$$\sigma_s = f_{yd} \tag{30}$$

And the steel area is given by:

$$A_s=\frac{F_s}{\sigma_s}=\frac{F_s}{f_{yd}} \tag{31}$$

The concrete strain ε_c in this case can be calculated by:

$$\frac{\varepsilon_c}{x}=\frac{\varepsilon_c+\varepsilon_{ud}}{d}\Rightarrow \varepsilon_c=\frac{x}{d}\left(\varepsilon_c+\varepsilon_{ud}\right) \tag{32}$$

Case 2: $x > x_{ult}$, as in shown in Figure 8 (c)

The concrete zone is at the critical strain, $\varepsilon_c = \varepsilon_{cu3}$ while $\varepsilon_s < \varepsilon_{ud}$. The steel strain ε_s can be calculated by:

$$\frac{\varepsilon_s}{d-x} = \frac{\varepsilon_{cu3} + \varepsilon_s}{d} \Rightarrow \varepsilon_s = \left(1 - \frac{x}{d}\right)\left(\varepsilon_{cu3} + \varepsilon_s\right) \quad (33)$$

In this case, it is not sure whether the steel works in full stress or not. The steel strain ε_s has to be checked if it is above or below the steel yield strain ε_{ys} as follows:

- **Case 2a: The steel works in full stress, $\varepsilon_s \geq \varepsilon_{ys}$**

The steel reinforcement works in full stress, above the yield strain and as a result the steel area can be calculated by Eq. (31).

- **Case 2b: The steel does not work in full stress, $\varepsilon_s < \varepsilon_{ys}$**

The steel reinforcement does not work in full stress, as it works below the yield strain. The steel stress σ_s can be calculated by Eq. (28) and the steel area can be calculated by Eq. (25). In this case, although the reinforcement area can be calculated, the design with only tensile reinforcement is not economic. Either compressive reinforcement should be considered, or an increase in the effective depth of the cross-section *d*.

6.1 Comment on the above cases

Eurocode 2 allows the designer to not limit the ultimate strain for steel when a horizontal top branch is assumed for the stress-strain diagram for steel. In this case, the concrete zone is assumed to be at the ultimate strain at all times at the ULS of the section under any design loads and the steel strain can take any value, without any limitation. If this is the case, then in the above investigation we have to set $x_{ult}=0$ and we have always the case of Figure 8(c) where the concrete zone is at the ultimate strain for all design cases.

This assumption can make things simpler and the calculations much easier. For reasons of completeness, in the present study we will continue to assume that the ultimate strain for steel is limited to ε_{ud}. This is done in order for the proposed methodology to be able to be extended also in the case where not a horizontal, but an inclined top branch is assumed for the steel stress-strain relation, with a maximum stress of $k \cdot f_{yk}/\gamma_s$ at ε_{uk}, where $k=(f_t/f_y)_k$. In this case Eurocode 2-Part 1-1 enforces the use of the strain limit of ε_{ud} for steel.

If the designer uses the proposed methodology and sets $\varepsilon_{ud}=\infty$, then we have the case of forcing the concrete zone to the ultimate strain and letting the steel take any strain value, as suggested by the Eurocode for the case of horizontal top branch stress-strain diagram. This will be illustrated also in the numerical examples section.

7 INVESTIGATION OF THE BI-LINEAR STRESS-STRAIN RELATION CASE

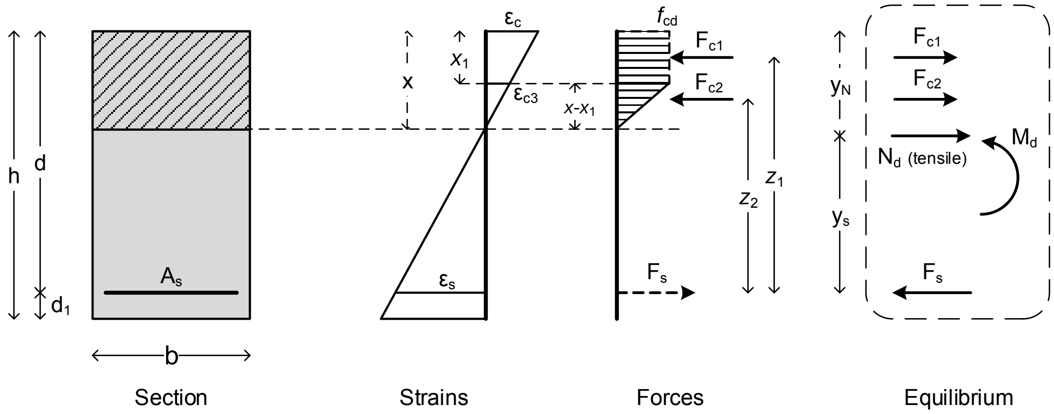

Figure 9: Cross section, strain and forces distribution and section equilibrium for the bi-linear stress-strain relation case, assuming $\varepsilon_c > \varepsilon_{c3}$.

The goal of the design is again to calculate the needed cross sectional area of steel reinforcement A_s. In order to calculate A_s, we need first to calculate the unknown quantities x, z_1 and z_2. We move again the external force N_d to the position of the steel reinforcement and we have the figure below.

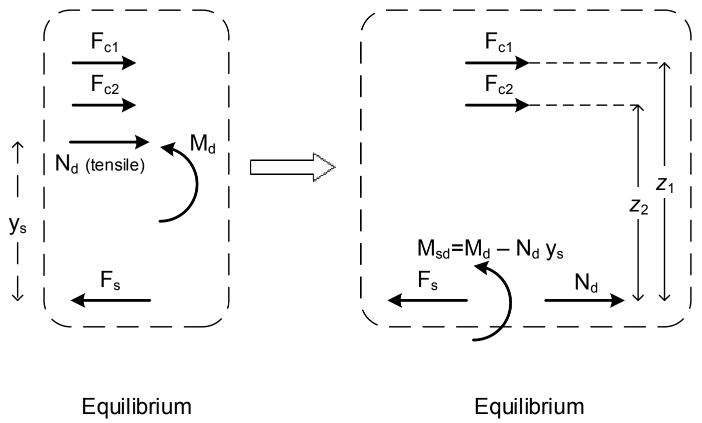

Figure 10: Equilibrium after moving the external force N_d to the position of the steel reinforcement.

We need to determine if at the ULS the concrete zone or the steel is at the critical strain. Again, first we put both materials at the ultimate strain, so we have:

$$\varepsilon_c = \varepsilon_{cu3} \tag{34}$$

$$\varepsilon_s = \varepsilon_{ud} \tag{35}$$

$$\frac{\varepsilon_{cu3}}{x} = \frac{\varepsilon_{ud} + \varepsilon_{cu3}}{d} = \frac{\varepsilon_{cu3} - \varepsilon_{c3}}{x_1} \Rightarrow \begin{cases} x = \dfrac{\varepsilon_{cu3}}{\varepsilon_{ud} + \varepsilon_{cu3}} \cdot d \\ x_1 = \dfrac{\varepsilon_{cu3} - \varepsilon_{c3}}{\varepsilon_{ud} + \varepsilon_{cu3}} \cdot d \end{cases} \tag{36}$$

$$F_{c1} = x_1 \cdot f_{cd} \cdot b \tag{37}$$

$$z_1 = d - \frac{x_1}{2} \tag{38}$$

$$F_{c2} = \frac{1}{2}\left(x - x_1\right) \cdot f_{cd} \cdot b \tag{39}$$

$$z_2 = d - x_1 - \frac{x - x_1}{3} = d - \frac{x + 2x_1}{3} \tag{40}$$

$$F_c = F_{c1} + F_{c2} \tag{41}$$

We will calculate the sum of moments at the steel reinforcement position. The sign of the sum of moments will show us whether the concrete zone or the steel is at the ultimate strain at the ULS. The sum of moments is (clockwise positive):

$$\Sigma M_{steel} = F_{c1} \cdot z_1 + F_{c2} \cdot z_2 - M_{sd} \tag{42}$$

We have then two cases:

Case 1. ΣM≥0
The concrete force has to be decreased for the equilibrium of the cross section. The steel stays at the ultimate strain ($\varepsilon_s = \varepsilon_{ud}$), while for concrete $\varepsilon_c \leq \varepsilon_{cu3}$, as shown in the figure below.

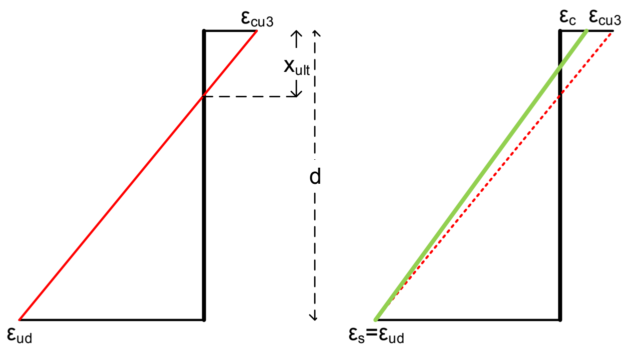

Figure 11: (b) Case 1: ΣM≥0, Steel at the ultimate strain.

Case 2. ΣM<0
The concrete force has to be increased for the equilibrium of the cross section. Concrete stays at the ultimate strain ($\varepsilon_c = \varepsilon_{cu3}$), while for steel $\varepsilon_s < \varepsilon_{ud}$, as shown in the figure below.

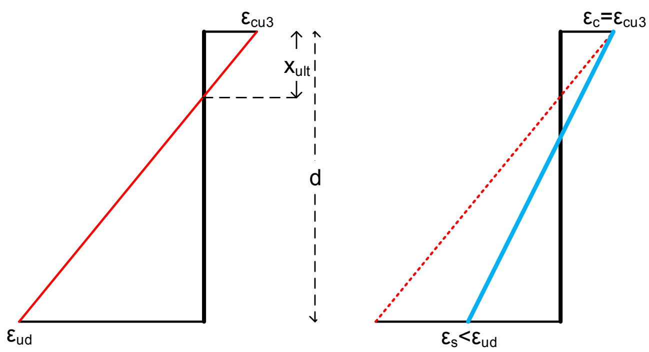

Figure 12: Case 2: ΣM<0, concrete zone at the ultimate strain.

For both cases, we need to determine the value of *x* that satisfies the equilibrium of the cross section. After having determined *x*, we can then proceed with the other calculations and finally end up with the needed reinforcement area A_s. The value of x can be determined by using trial and error iterations, or by using some kind of optimization in order to achieve section equilibrium. Good tools for this are MS Excel (Goal Seek or Solver functions) and also Matlab with its built-in root-finding and optimization tools. In the present study, we have used three equivalent approaches, (a) Solver function of MS Excel, (b) Matlab and (c) a home-made code which finds x by performing iterations, dividing the allowable height of the section by two at each iteration until convergence (equilibrium). All three approaches provide the same results at the end, as expected.

In the next sections, we will assume a value for x and we will end up with the equilibrium equation, i.e. the sum of moments at the steel reinforcement position which has to be zero at the equilibrium.

Case 1. ΣM≥0, Steel at the ultimate strain

We assume an initial value for x and we use the following equations:

$$\varepsilon_s = \varepsilon_{ud} \tag{43}$$

$$\frac{\varepsilon_c}{x} = \frac{\varepsilon_{ud}}{d-x} \Rightarrow \varepsilon_c = \frac{x}{d-x} \cdot \varepsilon_{ud} \tag{44}$$

- **Case 1a: If $\varepsilon_c > \varepsilon_{c3}$**

In this case we have the triangular diagram plus a rectangular diagram for the concrete zone and the upmost fiber of the concrete section works at the ultimate stress f_{cd}. From the similar triangles we have:

$$\frac{x_1}{\varepsilon_c - \varepsilon_{c3}} = \frac{d}{\varepsilon_c + \varepsilon_s} \Rightarrow x_1 = \frac{\varepsilon_c - \varepsilon_{c3}}{\varepsilon_c + \varepsilon_s} \cdot d \tag{45}$$

$$F_{c1} = x_1 \cdot f_{cd} \cdot b \tag{46}$$

$$z_1 = d - \frac{x_1}{2} \tag{47}$$

$$F_{c2} = \frac{1}{2}\left(x - x_1\right) \cdot f_{cd} \cdot b \tag{48}$$

$$z_2 = d - x_1 - \frac{x - x_1}{3} = d - \frac{2x_1 + x}{3} \tag{49}$$

$$F_c = F_{c1} + F_{c2} \tag{50}$$

$$\Sigma M_{steel} = F_{c1} \cdot z_1 + F_{c2} \cdot z_2 - M_{sd} \tag{51}$$

After we reach the equilibrium ($\Sigma M_{\text{steel}}=0$), and given that the steel reinforcement works in full stress, above the yield strain, the steel area can be easily calculated by Eq. (31).

- **Case 1b: If $\varepsilon_c \leq \varepsilon_{c3}$**

In this case we have only the triangular diagram for the concrete zone, there is no rectangular part for the stresses and the upmost fiber of concrete works at stress $\sigma_c \leq f_{cd}$, as follows:

$$\sigma_c = \frac{\varepsilon_c}{\varepsilon_{c3}} \cdot f_{cd} \leq f_{cd} \tag{52}$$

$$F_{c2}=\frac{1}{2}x\cdot\sigma_c\cdot b \tag{53}$$

$$z_2=d-\frac{x}{3} \tag{54}$$

$$F_c=F_{c2} \tag{55}$$

$$\Sigma M_{steel}=F_{c2}\cdot z_2-M_{sd} \tag{56}$$

Again, after we reach the equilibrium ($\Sigma M_{\text{steel}}=0$), and given that the steel reinforcement works in full stress, above the yield strain, the steel area can be calculated by Eq. (31).

Case 2. ΣM<0, concrete zone at the ultimate strain
We assume an initial value for *x* and we use the following equations:

$$\varepsilon_c=\varepsilon_{cu3} \tag{57}$$

$$\frac{\varepsilon_{cu3}}{x}=\frac{\varepsilon_s}{d-x}=\frac{\varepsilon_{cu3}-\varepsilon_{c3}}{x_1}\Rightarrow\begin{cases}\varepsilon_s=\dfrac{d-x}{x}\cdot\varepsilon_{cu3}\\ x_1=\dfrac{\varepsilon_{cu3}-\varepsilon_{c3}}{\varepsilon_{cu3}+\varepsilon_s}\cdot d\end{cases} \tag{58}$$

$$F_{c1}=x_1\cdot f_{cd}\cdot b \tag{59}$$

$$z_1=d-\frac{x_1}{2} \tag{60}$$

$$F_{c2}=\frac{1}{2}\left(x-x_1\right)\cdot f_{cd}\cdot b \tag{61}$$

$$z_2=d-x_1-\frac{x-x_1}{3}=d-\frac{2x_1+x}{3} \tag{62}$$

$$F_c=F_{c1}+F_{c2} \tag{63}$$

$$\Sigma M_{steel}=F_{c1}\cdot z_1+F_{c2}\cdot z_2-M_{sd} \tag{64}$$

- **Case 2a: $\varepsilon_s \geq \varepsilon_{ys}$**

The steel reinforcement works in full stress, above the yield strain and as a result the steel area can be calculated by Eq. (31).

- **Case 2b: $\varepsilon_s < \varepsilon_{ys}$**

The steel reinforcement does not work in full stress, as it works below the yield strain. The steel stress σ_s can be calculated by Eq. (28) and the steel area can be calculated by Eq. (25).

8 INVESTIGATION OF THE PARABOLIC-RECTANGULAR STRESS-STRAIN RELATION CASE

For the parabolic-rectangle stress-strain relation case, we use the same methodology as in the case of the bilinear stress-strain relation. The only difference is the shape of the concrete stress distribution where the triangular section becomes now parabolic, and also the ultimate strain and the strain corresponding to the start of the rectangular section which become ε_{cu2} (instead of ε_{cu3}) and ε_{c2} (instead of ε_{c3}), respectively, as shown in the figure below.

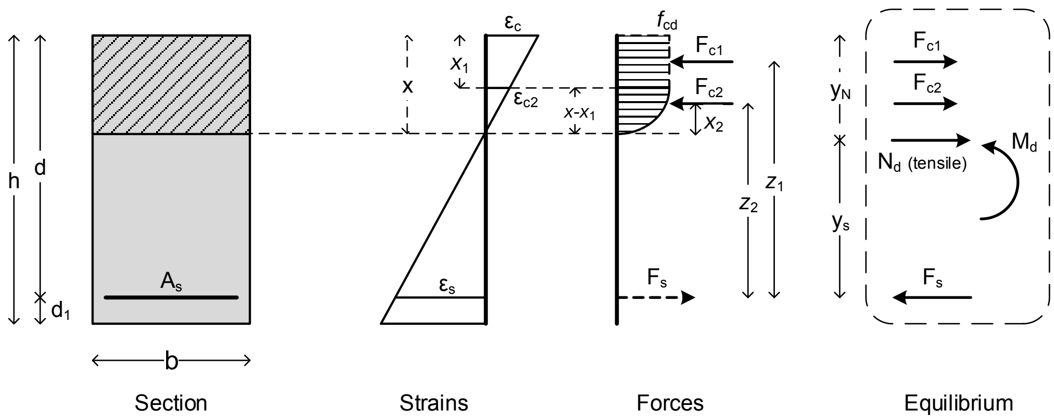

Figure 13: Cross section, strain and forces distribution and section equilibrium for the parabolic-rectangular stress-strain relation case, assuming $\varepsilon_c > \varepsilon_{c2}$.

In the above figure, x_2 is the distance from the neutral axis to the centroid of the parabolic section. The parabolic section is "full" in the figure, as $\varepsilon_c > \varepsilon_{c2}$. In the bi-linear case, the calculation of the area and centroid of the non-rectangular part was obvious, because of the triangular shape, but for the parabolic case, integration has to be used, as will be described in detail later.

Again, we need to determine if at the ULS the concrete zone or the steel is at the critical strain. First, we put both materials at the ultimate strain, so we have:

$$\varepsilon_c = \varepsilon_{cu2} \tag{65}$$

$$\varepsilon_s = \varepsilon_{ud} \tag{66}$$

$$\frac{\varepsilon_{cu2}}{x} = \frac{\varepsilon_{ud} + \varepsilon_{cu2}}{d} = \frac{\varepsilon_{cu2} - \varepsilon_{c2}}{x_1} \Rightarrow \begin{cases} x = \dfrac{\varepsilon_{cu2}}{\varepsilon_{ud} + \varepsilon_{cu2}} \cdot d \\ x_1 = \dfrac{\varepsilon_{cu2} - \varepsilon_{c2}}{\varepsilon_{ud} + \varepsilon_{cu2}} \cdot d \end{cases} \tag{67}$$

$$F_{c1} = x_1 \cdot f_{cd} \cdot b \tag{68}$$

$$z_1 = d - \frac{x_1}{2} \tag{69}$$

The above equations are almost the same as the ones used in the bi-linear case, but of course in the parabolic-rectangle case we use ε_{c2} and ε_{cu2} instead of ε_{c3} and ε_{cu3}. Yet, this time in order to calculate F_{c2} we need to integrate Eq. (2) to calculate the area of the parabolic part. For the parabolic part of the stress, i.e. for strains ε_c in the region $[0, \varepsilon_{c2}]$, we have the indefinite integral:

$$\int \sigma_c d\varepsilon_c = \int f_{cd} \left[1 - \left(1 - \frac{\varepsilon_c}{\varepsilon_{c2}} \right)^n \right] d\varepsilon_c = \varepsilon_c f_{cd} + \frac{\varepsilon_{c2} f_{cd} \left(1 - \dfrac{\varepsilon_c}{\varepsilon_{c2}} \right)^{n+1}}{n+1} \tag{70}$$

Thus the area E_1 of the full parabolic part [0, ε_{c2}] is given by the definite integral:

$$E_1 = \int_0^{\varepsilon_{c2}} \sigma_c d\varepsilon_c = \frac{n}{n+1}\varepsilon_{c2} f_{cd} \tag{71}$$

The area E_1 of the full parabolic part is shown in the figure below in black color.

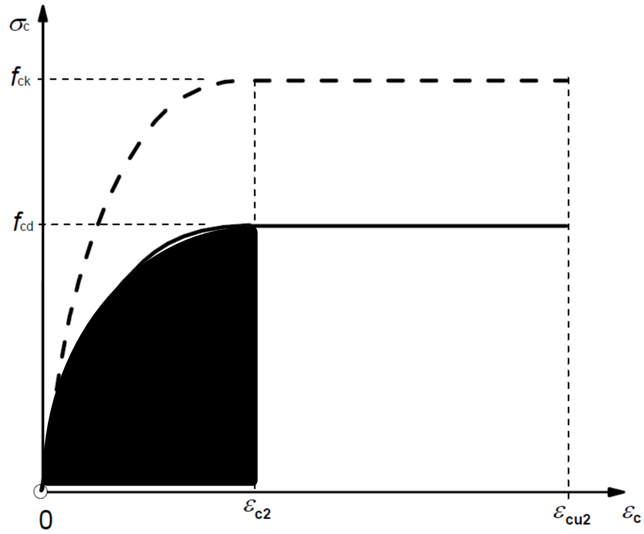

Figure 14: Area E_1 of the full parabolic part (for strains up to ε_{c2}) in black color.

If the integration is done on the cross section height, for the strain ε_{c2} the corresponding height of the section is (x-x_1) and as a result the corresponding area of the full parabolic part A_1 is given by:

$$A_1 = \frac{n}{n+1}(x - x_1) f_{cd} \tag{72}$$

The area A_1 of the full parabolic part is shown in the figure below in black color.

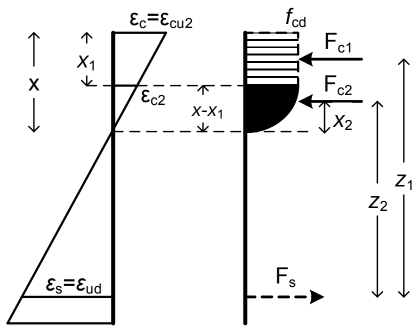

Figure 15: Strain and forces distribution.
The area A_1 of the full parabolic part is shown in black color.

The concrete force F_{c2} is given by:

$$F_{c2} = A_1 \cdot b = \frac{n}{n+1}(x - x_1) f_{cd} \cdot b \tag{73}$$

In order to calculate z_2 we need to calculate the distance x_2 defining the centroid of the A_1 area. In terms of strains, the centroid $\varepsilon_{centroid}$ of the E_1 area is given by the definite integral:

$$\varepsilon_{centroid1} = \frac{1}{E_1} \int_0^{\varepsilon_{c2}} \varepsilon_c \sigma_c d\varepsilon_c \tag{74}$$

The indefinite integral in this case is given by:

$$\int \varepsilon_c \sigma_c \mathrm{d}\varepsilon_c = \int \varepsilon_c f_{cd} \left[1 - \left(1 - \frac{\varepsilon_c}{\varepsilon_{c2}} \right)^n \right] \mathrm{d}\varepsilon_c = \frac{\varepsilon_c^{\ 2} f_{cd}}{2} + \frac{f_{cd} \left(\varepsilon_{c2} - \varepsilon_c \right)^{n+1} \left(\varepsilon_{c2} + \varepsilon_c \left(n+1 \right) \right)}{\varepsilon_{c2}^{\ n} \left(n+2 \right) \left(n+1 \right)} \tag{75}$$

Thus the centroid of the full parabolic part is given by:

$$\varepsilon_{centroid1} = \frac{1}{E_1} \int_0^{\varepsilon_{c2}} \varepsilon_c \sigma_c \mathrm{d}\varepsilon_c = \frac{n+3}{2(n+2)} \cdot \varepsilon_{c2} \tag{76}$$

If the integration is done on the section height, for the strain ε_{c2} the corresponding height of the cross section is (x-x_1) and as a result the corresponding centroid of the full parabolic part x_2 is given by:

$$x_2 = \frac{\varepsilon_{centroid1}}{\varepsilon_{c2}} \cdot (x - x_1) = \frac{n+3}{2(n+2)} \cdot (x - x_1) \tag{77}$$

Then we have

$$z_2 = d - x + x_2 \tag{78}$$

$$F_c = F_{c1} + F_{c2} \tag{79}$$

Again, we will calculate the sum of moments at the steel reinforcement position. The sign of the sum of moments will show us whether the concrete zone or the steel is at the ultimate strain at the ULS. The sum of moments is (clockwise positive):

$$\Sigma M_{steel} = F_{c1} \cdot z_1 + F_{c2} \cdot z_2 - M_{sd} \tag{80}$$

We then have again two cases:

Case 1. ΣM≥0

The concrete force has to be decreased for the equilibrium of the cross section. The steel stays at the ultimate strain (ε_s=ε_{ud}), while ε_c≤ε_{cu2}.

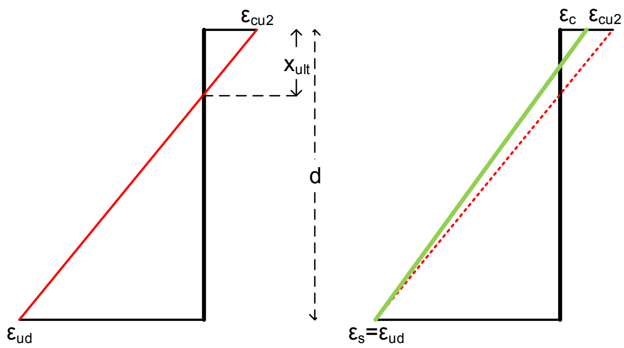

Figure 16: (b) Case 1: ΣM≥0, Steel at the ultimate strain.

Case 2. ΣM<0

The concrete force has to be increased for the equilibrium of the cross section. The concrete stays at the ultimate strain (ε_c=ε_{cu2}), while ε_s<ε_{ud}.

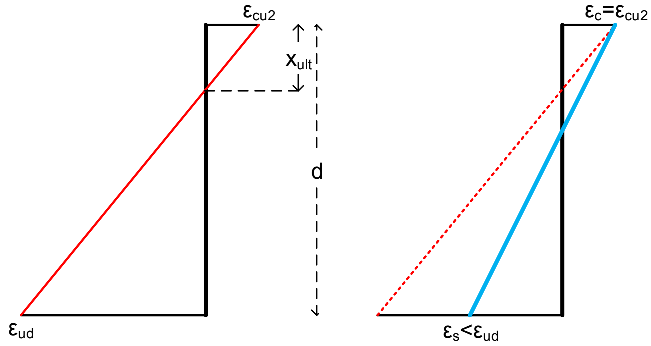

Figure 17: Case 2: ΣM<0, concrete zone at the ultimate strain.

The methodology is exactly the same as the one of the bi-linear case. To start, we assume a value for x and we should change it until we reach the final equilibrium. The equations below end up with the calculation of the sum of moments which has to be zero at the equilibrium.

Case 1. ΣM≥0, Steel at the ultimate strain

We assume an initial value for x and we use the following equations:

$$\varepsilon_s = \varepsilon_{ud} \tag{81}$$

$$\frac{\varepsilon_c}{x} = \frac{\varepsilon_{ud}}{d-x} \Rightarrow \varepsilon_c = \frac{x}{d-x} \cdot \varepsilon_{ud} \tag{82}$$

- **Case 1a: If ε_c>ε_{c2}**

In this case we have the parabolic diagram plus a rectangular diagram and the upmost fiber of concrete works at the ultimate stress f_{cd}. From the similar triangles we have:

$$\frac{x_1}{\varepsilon_c - \varepsilon_{c3}} = \frac{d}{\varepsilon_c + \varepsilon_s} \Rightarrow x_1 = \frac{\varepsilon_c - \varepsilon_{c3}}{\varepsilon_c + \varepsilon_s} \cdot d \tag{83}$$

$$F_{c1} = x_1 \cdot f_{cd} \cdot b \tag{84}$$

$$z_1 = d - \frac{x_1}{2} \tag{85}$$

In a similar way as previously (integrations), and since we have again a full parabolic part, we have:

$$F_{c2} = \frac{n}{n+1}(x - x_1)\, f_{cd} \cdot b \tag{86}$$

$$x_2 = \frac{n+3}{2(n+2)} \cdot (x - x_1) \tag{87}$$

$$z_2 = d - x + x_2 \tag{88}$$

$$F_c = F_{c1} + F_{c2} \tag{89}$$

$$\Sigma M = F_{c1} \cdot z_1 + F_{c2} \cdot z_2 - M_{sd} \tag{90}$$

After we reach the equilibrium (ΣM=0), and given that the steel reinforcement works in full stress, above the yield strain, the steel area can be easily calculated by Eq. (31).

- **Case 1b: If $\varepsilon_c \leq \varepsilon_{c2}$**

In this case we have only part of the parabolic diagram, there is no rectangular diagram and the upmost fiber of concrete works at stress $\sigma_c \leq f_{cd}$.

$$\sigma_c = \frac{\varepsilon_c}{\varepsilon_{c2}} \cdot f_{cd} \leq f_{cd} \tag{91}$$

This time in order to calculate F_{c2} we need to integrate Eq. (2) to calculate the area of the parabolic part, not for the full parabola (up to ε_{c2}), but for the region [0, ε_c] where $\varepsilon_c \leq \varepsilon_{c2}$. Using the indefinite integral of Eq. (70) we can calculate the corresponding area E_2 of the parabolic part for the region [0, ε_c] where $\varepsilon_c \leq \varepsilon_{c2}$ as a definite integral as follows:

$$E_2 = \int_0^{\varepsilon_c} \sigma_c d\varepsilon_c = f_{cd}\varepsilon_c + \frac{\varepsilon_{c2} f_{cd}\left(\left(1-\frac{\varepsilon_c}{\varepsilon_{c2}}\right)^{n+1} - 1\right)}{n+1} = f_{cd}\left(\varepsilon_c - \frac{\varepsilon_{c2}\left(1-\left(1-\frac{\varepsilon_c}{\varepsilon_{c2}}\right)^{n+1}\right)}{n+1}\right) \tag{92}$$

The area E_2 of the parabolic part for the region [0, ε_c] is shown in the figure below in black color.

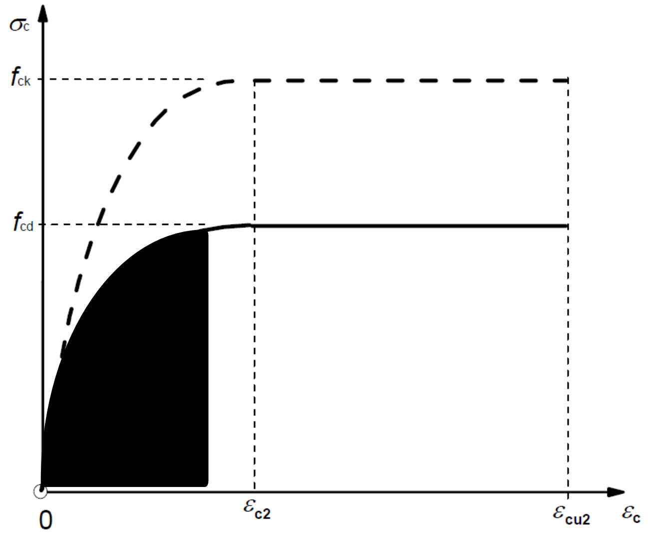

Figure 18: Area E_2 of the parabolic part for the region [0, ε_c] where $\varepsilon_c < \varepsilon_{c2}$, in black color.

If the integration is done on the section height, for a strain $\varepsilon_c < \varepsilon_{c2}$ the corresponding height of the cross section is x while for the theoretical strain ε_{c2} the corresponding height of the cross section would be $x \cdot \varepsilon_{c2}/\varepsilon_c$ and as a result the corresponding area of the parabolic part A_2 is given by:

$$A_2 = f_{cd}\left(x - \frac{\frac{\varepsilon_{c2}}{\varepsilon_c} x \left(1 - \left(1 - \frac{\varepsilon_c}{\varepsilon_{c2}} \right)^{n+1} \right)}{n+1} \right) = f_{cd} \cdot x \left(1 - \frac{\frac{\varepsilon_{c2}}{\varepsilon_c} \left(1 - \left(1 - \frac{\varepsilon_c}{\varepsilon_{c2}} \right)^{n+1} \right)}{n+1} \right) \quad (93)$$

The area A_2 of the parabolic part in this case is shown in the figure below in black color.

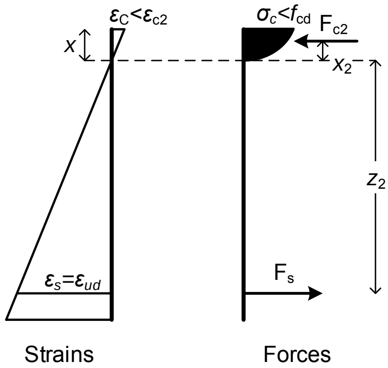

Figure 19: Strains and forces distribution.
The area A_2 of the parabolic part (for strains up to ε_c<ε_{c2}) is shown in black color.

The concrete force F_{c2} is given by:

$$F_{c2} = A_2 \cdot b = f_{cd} \cdot x \cdot b \left(1 - \frac{\frac{\varepsilon_{c2}}{\varepsilon_c} \left(1 - \left(1 - \frac{\varepsilon_c}{\varepsilon_{c2}} \right)^{n+1} \right)}{n+1} \right) \quad (94)$$

The calculation of z_2 for this case is the most difficult part. In order to calculate z_2 we need to calculate the distance x_2 defining the centroid of the A_2 area, as shown in Figure 19. In terms of strains, the centroid $\varepsilon_{centroid2}$ of the E_2 area is given by the following formula:

$$\varepsilon_{centroid2} = \frac{1}{E_2} \int_0^{\varepsilon_c} \varepsilon_c \sigma_c d\varepsilon_c \quad (95)$$

The definite integral $\int_0^{\varepsilon_c} \varepsilon_c \sigma_c d\varepsilon_c$ can be calculated from the indefinite integral of Eq. (75) as follows:

$$\int_0^{\varepsilon_c} \varepsilon_c \sigma_c d\varepsilon_c = \frac{\varepsilon_c^{\,2} f_{cd}}{2} + \frac{f_{cd} \left(\varepsilon_{c2} - \varepsilon_c \right)^{n+1} \left(\varepsilon_{c2} + \varepsilon_c \left(n+1 \right) \right) - f_{cd} \varepsilon_{c2}^{\,n+2}}{\varepsilon_{c2}^{\,n} \left(n+2 \right)\left(n+1 \right)} \quad (96)$$

If the integration is done on the section height, for the strain ε_c the corresponding height of the cross section is x and as a result the corresponding centroid of the full parabolic part x_2 is given by:

$$x_2 = \frac{\varepsilon_{centroid2}}{\varepsilon_c} \cdot x \tag{97}$$

Then we have

$$z_2 = d - x + x_2 \tag{98}$$

$$F_c = F_{c2} \tag{99}$$

Again, we will calculate the sum of moments at the steel reinforcement position. The sign of the sum of moments will show us whether the concrete zone or the steel is at the ultimate strain at the ULS. The sum of moments is (clockwise positive):

$$\Sigma M_{steel} = F_{c2} \cdot z_2 - M_{sd} \tag{100}$$

Case 2. ΣM<0, concrete zone at the ultimate strain

We assume an initial value for *x* and we use the following equations:

$$\varepsilon_c = \varepsilon_{cu2} \tag{101}$$

$$\frac{\varepsilon_{cu2}}{x} = \frac{\varepsilon_s}{d-x} = \frac{\varepsilon_{cu2}+\varepsilon_s}{d} = \frac{\varepsilon_{cu2}-\varepsilon_{c2}}{x_1} \Rightarrow \begin{cases} \varepsilon_s = \dfrac{d-x}{x} \cdot \varepsilon_{cu2} \\ x_1 = \dfrac{\varepsilon_{cu2}-\varepsilon_{c2}}{\varepsilon_{cu2}+\varepsilon_s} \cdot d \end{cases} \tag{102}$$

$$F_{c1} = x_1 \cdot f_{cd} \cdot b \tag{103}$$

$$z_1 = d - \frac{x_1}{2} \tag{104}$$

$$F_{c2} = \frac{n}{n+1}\left(x - x_1\right) f_{cd} \cdot b \tag{105}$$

$$x_2 = \frac{n+3}{2\left(n+2\right)} \cdot \left(x - x_1\right) \tag{106}$$

$$z_2 = d - x + x_2 \tag{107}$$

$$F_c = F_{c1} + F_{c2} \tag{108}$$

$$\Sigma M = F_{c1} \cdot z_1 + F_{c2} \cdot z_2 - M_{sd} \tag{109}$$

- **Case 2a: $\varepsilon_s \geq \varepsilon_{ys}$**

The steel reinforcement works in full stress, above the yield strain and as a result the steel area can be calculated by Eq. (31).

- **Case 2b: $\varepsilon_s < \varepsilon_{ys}$**

The steel reinforcement does not work in full stress, as it works below the yield strain. The steel stress σ_s can be calculated by Eq. (28) and the steel area can be calculated by Eq. (25).

9 NUMERICAL RESULTS

Four concrete sections will be examined in total. All three approaches for the stress-strain relations of concrete for the design of cross sections will be examined:

- Rectangular stress distribution
- Bi-linear stress-strain relation
- Parabola-rectangle diagram

Below are the common properties for all numerical examples:

- Steel class B500C (f_{yk}=500 MPa)
- E_s=200 GPa
- γ_c=1.50, γ_s=1.15
- ε_{uk}=75 ‰
- $\varepsilon_{ud}=0.9\cdot\varepsilon_{uk}$
- a_{cc}=1

9.1 Numerical example 1

The section of the first numerical example has the following properties:

- Concrete class C20/25
- Height h=50 cm
- Width b=25 cm
- d_1=5 cm
- M_d=60 kNm
- N_d=0

Below are the results of the design, for each of the three approaches for the stress-strain relations of concrete.

	Rectangular	**Bilinear**	**Parabola-rectangle**
A_s (cm^2)	3.22	3.22	3.22
ε_c (‰)	3.5	3.5	3.5
ε_s (‰)	26.5	24.6	26.8
x (m)	0.052	0.056	0.052
Critical material	Concrete	Concrete	Concrete

Table 3. Design results for the 1st numerical example.

It is clear that all three approaches give the same final result for the needed steel reinforcement area. Only minor differences can be found in the strains and the concrete zone height x.

9.2 Numerical example 2

The section of the second numerical example has the following properties:

- Concrete class C30/37
- Height h=60 cm
- Width b=30 cm
- d_1=5 cm
- M_d=100 kNm
- N_d=50 kN
- $y_N=h/2$

Below are the results of the design, for each of the three approaches for the stress-strain relations of concrete.

	Rectangular	**Bilinear**	**Parabola-rectangle**
A_s (cm^2)	4.90	4.91	4.90
ε_c (‰)	3.5	3.5	3.5
ε_s (‰)	53.1	49.6	53.8
x (m)	0.034	0.036	0.034
Critical material	Concrete	Concrete	Concrete

Table 4. Design results for the 2nd numerical example.

The three approaches give again almost the same final result for the needed steel reinforcement area. Again minor differences can be found in the strains and the concrete zone height x.

9.3 Numerical example 3

The section of the third numerical example has the following properties:

- Concrete class C70/85
- Height h=70 cm
- Width b=30 cm
- d_1=5 cm
- M_d=150 kNm
- N_d=100 kN
- y_N=h/2

Below are the results of the design, for each of the three approaches for the stress-strain relations of concrete.

	Rectangular	**Bilinear**	**Parabola-rectangle**
A_s (cm^2)	6.60	6.60	6.60
ε_c (‰)	2.1	2.4	2.3
ε_s (‰)	67.5	67.5	67.5
x (m)	0.020	0.023	0.023
Critical material	Steel	Steel	Steel

Table 5. Design results for the 3rd numerical example.

Again, the results are the same for all three cases. This time the steel is the critical material (at the ultimate strain) at the section equilibrium. The results above are calculated assuming a horizontal top branch for the steel stress-strain relation, but also limiting the maximum strain to ε_{ud}. Eurocode 2 allows the designer to not limit the ultimate strain for steel when a horizontal top branch is assumed for the stress-strain diagram for steel. In this case, the concrete zone is assumed to be at the ultimate strain at all times at the ULS and the steel strain can take any value, without any limitation. If we set $\varepsilon_{ud}=\infty$ (a very big number), then we have the results of the following table.

	Rectangular	**Bilinear**	**Parabola-rectangle**
A_s (cm^2)	6.60	6.60	6.60
ε_c (‰)	2.7	2.7	2.7
ε_s (‰)	84.7	77.4	78.4
x (m)	0.020	0.022	0.021
Critical material	Concrete	Concrete	Concrete

Table 6. Design results for the 3rd numerical example – No limitation for the steel strain.

We see that there is no difference in the required area of reinforcement A_s for the two cases, the results are exactly the same and only the reported material strains change. Of course, this time the critical material is the concrete zone, not the steel reinforcement.

9.4 Numerical example 4

The section of the fourth numerical example has the following properties:

- Concrete class C30/37
- Height h=50 cm
- Width b=25 cm
- d_1=5 cm
- M_d=378 kNm
- N_d=0

Below are the results of the design, for each of the three approaches for the stress-strain relations of concrete.

	Rectangular	**Bilinear**	**Parabola-rectangle**
A_s (cm^2)	26.14	33.78	26.67
ε_c (‰)	3.5	3.5	3.5
ε_s (‰)	2.14	1.69	2.13
x (m)	0.279	0.304	0.280
Critical material	Concrete	Concrete	Concrete
$\boldsymbol{\varepsilon_s/\varepsilon_{ys}}$	**0.98**	**0.78**	**0.98**

Table 7. Design results for the 4rd numerical example.

This time the results are different and in the bilinear case the difference is big. Taking a careful look at the results we can see that in this example, for all cases, $\varepsilon_s<\varepsilon_{ys}$ which means that the steel reinforcement works below the yield strain and as a result the steel cannot work in its full potential (f_{yd} stress). These are cases where the design is poor and not economic and the designer should either increase the height of the section or add compressive reinforcement, also.

In such cases with $\varepsilon_s<\varepsilon_{ys}$, the exact value of the steel strain ε_s is significant in the calculation of the final required reinforcement area as it affects directly the steel stress. In the bilinear case, ε_s is calculated as 1.69 ‰, lower than in the other two cases, and that affects the required reinforcement which is 33.78 cm^2, much more than in the other two cases.

The difference is big, but these cases are theoretical since in practical cases we would never design a section in such a way that the steel reinforcement would work below the yield strain.

10 CONCLUSIONS

- Eurocode 2-Part 1-1 gives us new tools in order to design concrete cross sections. Three approaches may be used for the stress-strain relation of concrete and another two approaches for the stress-strain relation of steel reinforcement. The simplest approach for concrete is the use of the Rectangular stress distribution. The other two approaches use the Bi-linear stress-strain relation and the Parabola-rectangle diagram, respectively.
- This paper presents a detailed methodology for the design of rectangular cross sections with tensile reinforcement, for all the three cases and for all concrete classes, covering all concrete classes, from C12/15 to C90/105. The methodology is general and all other Eu-

rocode parameters, such as γ_c, γ_s, a_{cc}, and others can be adjusted according to the preferences of the designer, without any limitation.

- The three approaches for concrete give almost the same results with each other for all "normal" cases examined. The differences are very slight and not significant from an engineering point of view. Big differences may occur in some "abnormal" cases where the effective moment is big for the section and as a result the steel reinforcement works below the yield strain ε_{ys}. In any case, these cases have to do with bad section design and they should be avoided. The best solution for these cases is to add height to the concrete section, and/or add compressive reinforcement also.
- Eurocode 2 allows the designer to not limit the ultimate strain for steel when a horizontal top branch is assumed for the stress-strain diagram for steel. In this case, the concrete zone is assumed to be at the ultimate strain at all times at the ULS and the steel strain can take any value, without any limitation. In the proposed methodology this can be achieved by setting $\varepsilon_{ud}=\infty$ (a very big number) for the allowed steel strain. This was investigated in a numerical example where the steel was the critical material and the result showed that it made no difference in the final steel reinforcement area.
- A more detailed investigation has to be made regarding the three stress-strain approaches for concrete in order to check if there are cases where the three approaches can lead to different results. The next research step should be to use the proposed methodology in order to generate dimensionless charts showing the required reinforcement for any loading and any section. In this way, a general direct comparison of the three cases can be performed.

REFERENCES

[1] EN 1992-1-1. Eurocode 2: Design of concrete structures. Part 1-1: General rules and rules for buildings. Brussels, 2004.

[2] B. Rosca, M. Petru, Reinforced Concrete Section Design to Bending according to EN 1992-1-1/2004 – Eurocode 2. *Buletinul Institutului Politehnic din Iasi*, 53-66, 2009.

[3] E. Dulinskas, D. Zabulionis, Analysis of equivalent substitution of rectangular stress block for nonlinear stress diagram. *ISSN 1392 – 1207, Mechanika*, ***6 (68)***, 53-66, 2007.

[4] D. Zabulionis, E. Dulinskas, Analysis of compression zone parameters of cross-section in flexural reinforced concrete members according to EC2 and STR 2.05.05. *ISSN 1392 – 1207, Mechanika*, ***3 (71)***, 12-19, 2008.

[5] I. Židonis, Strength calculation method for cross-section of reinforced concrete flexural member using curvilinear concrete stress diagram of EN-2. *11th International Conference on Modern Building Materials, Structures and Techniques (MBMST 2013)*, Vilnius, Lithuania, May 16 – 17, 2013.

[6] I. Židonis, A simple-to-integrate formula of stress as a function of strain in concrete and its description procedure. *ISSN 1392 – 1207, Mechanika,* ***4(66)****, 23-30, 2007.*